Network Structures and Joint Liability Contracts: Authoritarian vs Egalitarian Networks

Bahar Rezaei

Anderson School of Management, University of California, Los Angeles, 110 Westwood Plaza, Los Angeles, CA 90095, USA; rezaei@ucla.edu

Sriram Dasu

Marshall School of Business, University of Southern California, Los Angeles, CA 90089, USA; dasu@marshall.usc.edu

Reza Ahmadi

Anderson School of Management, University of California, Los Angeles, 110 Westwood Plaza, Los Angeles, CA 90095, USA; reza.ahmadi@anderson.ucla.edu

Abstract

(1) Problem definition: We study sustainability of group lending under joint liability contracts when borrowers are subject to strategic default. The benevolent lender optimizes the borrower welfare while covers her costs of lending, and social ties between borrowers serve as monitoring tools.

(2) Academic/Practical relevance: To the best of our knowledge, our work is the first theoretical study of this problem that models borrowers social network structures as patterns of network monitoring.

(3) Methodology: The methodology employed in this paper is reverse game theory or mechanism design.

(4) Results: The results show that a higher average degree of centrality and connectivity in the borrower network allows for more favorable contract terms (i.e., a higher loan ceiling and a lower repayment amount). Accordingly, a borrower network with a complete structure qualifies for the best contract terms. Intuitively, a complete network, which provides full monitoring, induces the highest level of cooperation among borrowers and guarantees the highest level of group performance (i.e. a higher chance of game continuation and a higher repayment rate). Further, we argue that while for smaller groups, the highly decentralized ring networks (egalitarian) may

have a higher performance than the highly centralized star networks (authoritarian), for larger groups, star networks outperform the rings. We also prove that enlarging the group can limitedly contribute to its performance and suggest an upper bound on the size of the borrower group.

(5) Managerial implications: The setting studied in this paper can emerge in a variety of team cooperation or social dilemma contexts that deal with monitoring, and thus the results are valid wherever the interests of the individual and of the group are not aligned and agents exert externalities on one another.

Keywords: Joint Liability, Strategic Default, Monitoring, Social Networks, Graph Theory

1 Introduction

Microfinance Institutions (MFIs) leverage borrowers' social capital and local information to lend to borrowers that are otherwise unbankable. This type of credit lending has long attracted the attention of scholars, governments, and practitioners for its success in lending to the poor when the conventional collateral and well-functioning formal institutions are missing. One of the most studied and practiced lending models used by MFIs is lending to self-selected groups of borrowers under joint liability contract.[1] In a jointly liable group, if one member does not repay his loan, the other members are held responsible to repay for him. The joint liability lending scheme is believed to improve repayment rates in two ways: it incentivizes borrowers not only to help their group members repay their installments but also to avoid delinquent behavior that may trigger social sanctions (Besley and Coate, 1995). Microfinance literature argues that joint liability creates an effective way of screening, monitoring and enforcement of contracts (see e.g., Ghatak, 1999; and Armendáriz and Morduch, 2010).

In this paper, we analyze the following microcredit lending game. A benevolent lender (she) provides loans to a self-selected group of micro-entrepreneur borrowers with joint liability. Then the loans are invested into projects that are disjoint and have an equal chance of success. Note that the social network theory literature believes that people tend to homophilous, meaning that they tend to have stronger ties with people who are similar to themselves (Borgatti and Halgin, 2011).

[1]Recently, some of older MFIs have shifted from joint liability to individual lending. Nevertheless, group lending with joint liability remains the main channel for microcredit lending by most MFIs. Moreover, MFIs moving away from explicit joint liability have retained some of its traditions such as organizing borrowers into groups and holding group meetings on a regular basis (Haldar and Stiglitz, 2016; and De Quit et al., 2018).

Thus, it is not unrealistic to assume that the chance of project success for all members of a self-selected group is equal. We also assume that borrowers can monitor each others' project output at least to some extent, but the lender cannot. This is also a reasonable assumption considering that group members are self-selected and have better information about each other than the lender. The entire group will be deprived from future loans if the total repayment obligation is not met. Such a setting creates a risk-sharing arrangement that is subject to ex post moral hazard: a borrower who is expected to repay may prefer not to.

We model the foregoing situation as a dynamic infinitely repeated game. Borrowers receive individual contracts, specifying the amounts of loan L and repayment R. The contracts are equal for all group members. Borrowers invest their loans on projects with disjoint return. All projects have the same chance of success and yield either a high Y^H or low $Y^L = 0$ return. After the realization of project outputs, each successful borrower decides to repay his loan or default while unsuccessful borrowers are not supposed to repay. We discriminate between two types of default: *strategic default*, in which borrowers are not willing to repay they have high project return, and *non-strategic default*, in which borrowers do not repay because they have little or no project returns. Members of the group can identify strategic default at least to some extent, but the lender cannot. The lender delegates monitoring to group members by depriving the entire defaulting group from future loans and thus decreases incentives for strategic default. We assume that, in the game played between group members, they play a grim-trigger strategy. Meaning that, they stop repaying their loans, as well as the loans of their defaulting peers, as soon as someone defaults strategically.[2] In such a case, no further loans are given to the group.

We model the borrower's social network as a set of pre-existing relationships, such as friendships or family ties, in which network links serve as a monitoring instrument ensuring that borrowers live up to their responsibilities. We assume that borrowers with a *strong* social tie can fully monitor the realized project output of each other while borrowers with a *weak* social tie can only partially monitor each other's project output. Having access to a higher level of monitoring allows a borrower to reveal his truthfulness and verify the truthfulness of his peers and thus receive and give support

[2]This is a plausible assumption as both real world experience and scientific evidence suggest that in social dilemma situations, many people have a strong aversion against being the "gullible". Fehr and Gächter (2000), for example, argue that "those who cooperate may be willing to punish free-riding, even if this is costly for them and even if they cannot expect future benefits from their punishment activities". For another example, see Carpenter (2007).

with a higher probability when needed. We also assume that indirect social ties, in which two members are strongly connected through a third member, are more reliable than weak ties and less reliable than strong ties. Such a setting allows us to investigate how variation in the structure of borrowers social network (who is close friend of whom) relates to their collective levels of cooperation and consequently to the lending outcome.

We derive the optimal joint liability contract depending on the structure of the borrowers' social network and explore how network structure affects both the chance of a successful group repayment and the expected repayment of each individual borrower, which in turn affect the joint liability contract terms. We first look at networks of four borrowers (henceforth tetrads), which are the smallest networks that allow for distinctive line, star, ring, core-periphery, and complete structures to develop intuition, and then we show that the intuitions extend over networks of $n \geq 5$ borrowers (henceforth polyads). In practice, the size of the joint liability groups varies usually between four and ten members (Ahlin, 2015).

While looking at larger networks of n borrowers, we focus on two specific structures: the highly centralized *star* structure, in which a core borrower has strong links with all other group members while the others are only indirectly connected to each other through the core borrower, and the highly decentralized *ring* structure, in which every borrower has two strong links and two strong indirect links with other group members. The reason for this focus, apart from simplifying the analysis, is as follows. Many theoretical studies provide evidence for the formation of ring and star structures (or convergence to these structures) as likely evolving structures in strategic settings. For example, in their seminal work, Bala and Goyal (2000) theoretically prove that the ring and the star structures are the only strict Nash equilibria of strategic network formation. Ring- and star-like topologies have also received strong support from empirical and experimental studies (see e.g., Falk and Kosfield, 2012; Goeree et al., 2009; Kumar et al., 2010; and Baker and Faulkner, 1993).

Our results show that a borrower network, in which all borrowers are connected through strong social ties, may be the best network to induce cooperation among borrowers and thus improve their welfare. Intuitively, borrower networks with full monitoring promote cooperation in two ways. When a group members defaults, all other members can monitor his state of project and thus would help him to repay his loan if he defaults non-strategically and would punish him if he defaults strategically. Although the best network structure that can induce cooperation is the complete network structure, in the real world, borrowers networks can be less than complete, especially in urban areas that people have less reliable information about their neighbors and friends. In such situations, it makes sense to look at the effect of different social network structures on the borrower's behavior and adjust the lending contracts accordingly.

The results also suggest that for tetrad borrower groups, higher the total degree of the network, higher its performance with one exception: the star structure may have a higher performance than the ring structure under some conditions. Moreover, removing a borrower of a lower degree from a tetrad network can benefit the network only when projects are not too risky. For polyad borrower groups, we extend this result and show that a higher average degree of centrality and connectivity improves the performance of the network both in terms of a higher chance of game continuation and a higher expected repayment (or robustness of the network). In other words, both the number and the patterns of the strong links in a network can affect the outcome of the lending game. Moreover, connected networks, in which every member is fully monitored by at least one other member of the group, have a higher performance than disconnected networks. Thus, any network with some strong connections is better off removing a borrower, which is poorly monitored by other group members except when projects are highly risky.

We prove that in larger groups, the star structure can maintain a higher chance of game continuation and a higher expected repayment than the same-sized ring structure, while in smaller groups, the ring structure dominates the star structure of the same size both in terms of a higher chance of project success and a higher expected repayment. Intuitively, being a member of either the highly centralized star structure or the highly decentralized ring structure has both benefits and costs. In a star group, all members can monitor each other's project outcomes at least indirectly, and thus they are able to verify truthfulness of their defaulting members and maintain a higher level of cooperation, but the group falls apart if the core member fails on his repayment. Although, in a ring group, no member is overly important in terms of robustness, some members are out of each other's monitoring scope and cannot tell apart strategic defaults from non-strategic defaults. In larger groups, the monitoring disadvantage of a ring structure with respect to a star structure intensifies as every member is able to monitor at most four other members. However, in small groups, the monitoring disadvantage of a ring structure fades away and its robustness advantage comes into play.

We further prove that a larger group size has two counteracting effect on the borrower welfare. On one hand, a large group can ceteris paribus provide a borrower with a higher chance of being paid off and continue receiving loan in the next round. On the other hand, a larger group can put a successful borrower in charge of repaying for many other defaulting members. We find an upper bound for the size of the group depending on the chance of project success and the borrower discount factor.

2 Literature Review

This research is inspired by the line of research that examines how social ties can be used to contain free-riding and induce cooperation in settings, in which formal contract enforcements are lacking, such as: cooperations in organizations (Inkpen and Tsang, 2005), cooperations between immigrant entrepreneurs (Kalnins and Chung, 2006), enforcement of informal contracts (Karlan et al., 2009), reciprocity (Leider et al., 2009), repayment of online peer-to-peer (Lin et al., 2013) or other informal loans (Canales and Greenberg, 2015).

Our paper contributes to the microfinance literature that views borrowers' social ties as collateral that can be used to circumvent moral hazard problem and encourage repayment (see e.g. Besley and Coate, 1995; Zeller, 1998; Ghatak and Guinnane, 1999; Wydick, 1999; Hermes et al., 2005; Cassar et al., 2007; Paal and Wiseman, 2011; Cason et al., 2012; Feigenberg et al., 2013). We differ from this literature by explicitly modeling social ties as means of peer monitoring. In our model, a stronger social tie between two borrowers allows them to better monitor each other's project outcome, which in turn induces cooperation because it provides a higher chance of both being backed by peers in case of non-strategic default and being punished by peers in case of strategic default. The extant microfinance literature provides evidence that stronger social ties between group members can improve the repayment rate. For example, Besley and Coate (1995) and Ghatak and Guinnane (1999) theoretically prove that high social ties deter group members shirking on repayments. Moreover, Zeller (1998) and Cason et al. (2012) empirically show stronger social ties lead to improved repayment rates and lender profitability if it exceeds monitoring costs. Our research compliments this literature by discussing that apart from the prevalence and the strength of the social ties in a borrowing group, the existing patterns of these ties (who can monitor whom) can also affect the borrowers cooperative behavior and thus the lending outcome.[3]

Another literature that is closely relevant for this paper is the literature that highlights the effect of social network structures on team cooperation and team performance. Examples of these researches include but are not limited to Rulke and Galaskiewicz (2000), Sparrowe et al., (2001), Reagans and Zuckerman (2001), Collins and Clark (2003), Cummings and Cross (2003), Carson et al. (2007), Gloor et al. (2008), and more recently Grund (2012). Rulke and Galaskiewicz (2000) find that decentralization of knowledge in top management teams is positively associated with group performance. Sparrowe et al., (2001) observed that social networks can affect individual and group performance both positively and negatively. The authors show that groups with decentralized communication structures outperform groups with centralized communication structures. Reagans and Zuckerman (2001) demonstrates that network density in corporate R&D teams is positively associated to network productivity. Collins and Clark (2003) show that social networks of top managers in high-technology firms mediate the relationships between the human resource practices and firm performance. In the context of leadership in a global organization, Cummings and Cross (2003) indicate that core-periphery and hierarchical group communication structures were negatively related with group performance. Carson et al. (2007) found that shared leadership in consulting teams predicts team performance. Gloor et al. (2008) provide evidence that balanced communication structures between online team members are positively related to team performance. Grund (2012) empirically studies the interaction network and performance of professional soccer teams in the English Premier League (EPL) and shows that networks with high intensity (controlling for interaction opportunities) and low centralization have a better team performance.

A farther relevant literature for this paper is the literature on risk-sharing in social networks. For prominent recent examples we could mention Bramoullé and Kranton (2007b), Attanasio et al. (2012), Ambrus et al. (2014), Acemoglu et al. (2016), and Mobius and Rosenblat (2016).

[3] A similar idea has been examined in the context of public good games by various researchers. For example, the experimental works of Carpenter and Kariv (2012) prove that connected network with less than complete structure can achieve a high level of cooperation; and Leibbrandt et al. (2015) find that the structure of the network significantly affects contributions and punishment decisions of its members.

3 Model

The model in this section builds on our previous work Rezaei et al. (2017). Consider an infinitely repeated lending game, with a benevolent lender and n borrowers. We assume that borrowers with a strong social tie can fully monitor each other's project returns while borrowers with a weak social tie can only partially monitor each other's project returns. Each period of the game has the following three steps.

1. Each borrower receives an individual contract (L, R), specifying the amount of loan L, which is enough to cover the costs of project, and the repayment $R \geq L(1+\epsilon)$ (primary loan plus interest).

2. Each borrower invests the entire L in his project that will either succeed with a chance of $\alpha \in [0, 1]$ and yields a high return $Y^H > 0$, which is increasing in the amount of loan L, or not succeed with a chance of $1 - \alpha$ and yields a zero return. Group members can at least partially monitor each other's project returns, but the lender cannot.

3. Borrowers simultaneously make their repayment decisions. If i members default, the other $n - i$ members will be asked to collectively pay an additional amount iR to the lender for their defaulting peers. If the total group repayment is equal to nR or more, the group receives future financing; otherwise, the lender will exclude the entire group from future loans.

These three steps are repeatedly played until the lender realizes that the borrowers are not entitled to financing next period. In periods in which no loan is received, borrowers' utility will be zero. We assume that projects do not differ in their riskiness (i.e., α is the same for all borrowers) and each borrower always invests in the same project.

Two types of defaults are possible: *strategic default,* in which the borrower does not repay although he had high outcome Y^H, and *non-strategic default* as a result of obtaining zero outcome resulting from bad luck. Although, the lender is unable to monitor whether a borrower defaults strategically or non-strategically, borrowers are at least partially able to monitor strategic defaults of their peers.

We assume borrowers continue cooperating until someone defaults. If a borrower j defaults strategically, borrowers play grim-trigger; that is, they stop paying for him. In this case, the group will not be eligible for further loans. If a borrower j defaults non-strategically, then a successful strongly connected friend (a strong or a first-hand tie) of j, who is aware of j's truthfulness, would pay for him; and a successful weakly connected friend of j, who cannot verify j's truthfulness, would pay for him only with probability β. Whether a successful strongly connected friend of friend (an indirect or a second-hand tie) of j would pay for him depends on the their friend in common; that is, an indirect tie would pay for j if their friend in common pays; otherwise, he would pay for him only with probability β.

The benevolent lender strives to maximize the payoff of each borrower contingent on the following criteria. First, each borrower must be willing to accept a loan (the repayment amount must be affordable); second, each successful borrower must have the incentive to repay for himself and for each defaulting peer (in the worst case that all other members default, he must still be willing to repay for the entire group); and third, the lender must break even, meaning that she must maintain a sustainable lending operation over the entire loan portfolio by charging the appropriate repayment amount.

Borrower Networks: Following the conventional social network theory literature, we employ a graph theoretic language to describe borrower networks. A network N consists of a set of nodes, $\mathcal{J} = \{1, 2, ..., j \geq 2\}$, representing borrowers, and a set of edges, $\mathcal{E}$, representing strong links between borrowers. In such a network, if $(i, k) \in \mathcal{E}$, then borrowers i and k are strongly connected. If $(i, k) \notin \mathcal{E}$, but there is a borrower f such that $(i, f) \in \mathcal{E}$ and $(f, k) \in \mathcal{E}$, then borrowers i and k are indirectly connected. Borrowers i and k are said to be weakly connected if they are neither strongly nor indirectly connected. We denote the number of borrowers in a group by $|\mathcal{J}| = j$ and the total number of strong links between borrowers by $|\mathcal{E}| = e$, which we also call the total degree of the network. We also denote a network with $j \geq 2$ members and e strong links by $N(j, e)$.

We assume that information achieved through indirect ties compared to the one through a direct tie is of lower value. This is a reasonable assumption as indirect monitoring can cause information to be distorted. We assume that the fidelity of the information diminishes as the length of the path grows. Each arc diminishes the fidelity by a factor σ. Accordingly, information transmitted over two strong arcs has a weight of $\sigma < 1$, and a path with $k + 1$ arcs has a weight of σ^k. Information between two weakly connected nodes has a weight of $\beta < \sigma$. For simplicity of our calculation, we assume that for any $k \geq 2$, $\sigma^k < \beta$.

In a network of n nodes, the *degree of centrality* of a node j is its sum of value of shortest paths to all other nodes divided by $n-1$, and the average degree of centrality of a network is the average, over all nodes, of their degrees of centrality. In our model, the shortest path between any two nodes can be of length 1, σ, or β. For example, in a ring tetrad network (i.e., a network of four nodes), the degree of centrality of each node and the average degrees of centrality of the network are $\frac{2+\sigma}{3}$. For more examples, see Figures 3 and 4.

In a network of n nodes, the *degree of connectivity* of a node j is the sum of value of links that the network loses if j is removed divided by $n-1$, and the average of connectivity of a network is the average of the degrees of connectivity of all nodes. For example, in a ring tetrad network, the degree of connectivity of each node and the average degree of connectivity of the network are $\frac{2+2\sigma}{3}$. For more examples, see Figures 3 and 4.

A network of $j \geq 2$ members is said to have

- an empty structure when $E = \{\}$,
- a star structure when $E = \{(1,j), (2,j), ..., (j-1,j)\}$,
- a ring structure when $E = \{(1,2), (2,3), ..., (j-1,j), (j,1)\}$,
- and a complete structure when $E = \{(1,2), (1,3), ..., (1,j), (2,3), (2,4), ..., (2,j), ... (j-1,j)\}$.

4 Joint Liability Contract

In this section, we characterize feasible joint liability contracts and examine how contract terms change in response to changes in the network structure. All group members receive the same contract.

The expected utility of a repaying borrower j with a discount factor δ in a period in which he obtains financing is

$$v_j^R = \alpha Y^H - k_j R + \delta Q v_j^R,$$

where $0 \leq Q \leq 1$ is the probability of receiving a loan in the next round, and $k_j R$ $(0 \leq k_j \leq 1)$ is the borrower's expected repayment. The above equation can be rewritten as

$$v_j^R = \frac{\alpha Y^H - k_j R}{1 - \delta Q}.$$

Accordingly, the total expected utility of the network in a period in which borrowers obtains financing will be

$$V^R = \sum_{j=1}^{n} v_j^R = \frac{n\alpha Y^H - \sum_{j=1}^{n} k_j R}{1 - \delta Q}.$$

The benevolent lender seeks to maximize V^R and not her own profit, but she still needs to get her money back to sustain lending overtime. Thus, the lender has to design the lending contract such that the total expected repayment of the group covers at least the total loan granted to the borrowers and the costs of lending services; that is,

$$\sum_{j=1}^{n} k_j R \quad \geq \quad nL\left(1 + \epsilon\right).$$

In other words, the average expected repayment in a network $\frac{\sum_{j=1}^{n} k_j R}{n}$ should exceed a loan plus the costs of services. In the rest of the paper, KR denotes the average expected repayment in a network.

Moreover, the lender has to ensure that the total repayment is both affordable and better than default for a successful borrower; that is,

$$nR \leq Y^H,$$

$$Y^H - nR + \delta v_j^R \geq Y^H.$$

Note that if the latter constraint holds for the member with the lowest v_j^R in the network, then it holds for all members of the network. Since v_j^R is decreasing in $k_j R$, it would be sufficient to satisfy the constraint for the member with the highest $k_j R$.

Thus, the lender faces the following optimization problem,

$$\underset{L,R}{maximize} \quad V^R = \frac{n\alpha Y^H - nKR}{1-\delta Q} \tag{1}$$

$$s.t. \quad nR \leq Y^H \tag{2}$$

$$nR \leq \delta v_m^R \tag{3}$$

$$KR \geq L(1+\epsilon), \tag{4}$$

where v_m^R is the lifetime utility of the borrower with the highest expected repayment k_m in the network. Proposition 1 solves the lender's optimization problem and specifies the feasible contract terms for a network of n members depending on the structure of the borrower network captured by Q, K, k_m.

Proposition 1. *Joint liability is feasible if and only if*

$$L \leq \mathscr{L} = \begin{cases} \frac{KY^H}{n(1+\epsilon)} & \text{if} \quad \delta \geq \frac{n}{nQ+n\alpha-k_m} \\ \frac{\alpha\delta KY^H}{(n-n\delta Q+\delta k_m)(1+\epsilon)} & \text{if} \quad \delta \leq \frac{n}{nQ+n\alpha-k_m} \end{cases},$$

$$R = \frac{L(1+\epsilon)}{K}.$$

The total expected lifetime utility for borrowers receiving the joint liability contract (L, R) *will amount to* $V^R = \frac{n\alpha Y^H - nL(1+\epsilon)}{1-\delta Q}$.

Proposition 1 shows that the loan ceiling $\mathscr{L}$, and consequently the total borrower welfare V^R, is increasing in both K and Q, but decreasing in k_m. Intuitively, a better connected network has a higher chance of returning the loan and a higher total expected repayment and thus could be trusted to receive a larger loan. However, if a larger portion of the expected repayment of the network comes from one individual member, the default of that individual member would be difficult to handle for the rest of the network.

This proposition also highlights that the loan ceiling depends positively on the borrowers' discount factor, as $\frac{\alpha\delta KY^H}{(n-n\delta Q+\delta k_m)(1+\epsilon)}$ is strictly increasing and $\frac{KY^H}{n(1+\epsilon)}$ is constant in δ. Intuitively, if borrowers value receiving future loans more, they would have a higher incentive to repay their loans and thus could be trusted to repay larger loans. But there is a limit on the amount of loan they can be granted and the loan size cannot grow larger after some point.

The following examples provide some understanding about Q, K, and v_m^R. For an empty network of n members, in which all members are weakly connected, there are $\binom{n}{i}$ number of cases, in which i members default on their repayments and $n-i$ repay. Each successful member would repay for the defaulting members with probability β^i. Thus, for an empty network of n members, the chance of a successful group repayment Q^e will be

$$Q^e = \sum_{i=0}^{n-1} \binom{n}{i} \alpha^{n-i} (1-\alpha)^i \left(\beta^i\right)^{n-i}.$$

In an empty network of n members, every member j whose project succeeds is expected to repay R for himself and $\frac{i\beta R}{n-i}$ on behalf of his defaulting peers. Thus, for an empty network of n members, the average expected repayment $K^e R$ is

$$\begin{aligned} K^e R &= \frac{1}{n} \sum_{j=0}^{n} \alpha \sum_{i=0}^{n-1} \binom{n-1}{i} \alpha^{n-1-i} (1-\alpha)^i \left(R + \frac{i\beta R}{n-i}\right) \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} \alpha^{n-i} (1-\alpha)^i \left(1 + \frac{i\beta}{n-i}\right) R. \end{aligned}$$

Note that the highest expected repayment in an empty network is the same as the average expected repayment; i.e., $k_m^e = K^e$, and thus

$$v_m^R = \frac{\alpha Y^H - K^e R}{1 - \delta Q^e}.$$

For the empty network, the expected repayment is the same for all group members, but this may not be the case for other network structures. More precisely, depending on how many first-hand and second-hand connections of a repaying borrower are among the i defaulting members, some of the β's will be replaced by σ or 1.

For a complete network of n members, in which all borrowers are strongly connected, the chance of game continuation Q^c and the average expected repayment $K^c R$ of the network can be calculated by replacing β with 1 in the above formulas; that is,

$$\begin{aligned} Q^c &= \sum_{i=0}^{n-1} \binom{n}{i} \alpha^{n-i} (1-\alpha)^i \\ &= [1-(1-\alpha)^n], \\ K^c R &= \sum_{i=0}^{n-1} \binom{n-1}{i} \alpha^{n-i} (1-\alpha)^i \left(1+\frac{i}{n-i}\right) R \\ &= [1-(1-\alpha)^n] R. \end{aligned}$$

For such a network, $k_m^c = K^c$, and thus

$$v_m^R = \frac{\alpha Y^H - K^c R}{1-\delta Q^c}.$$

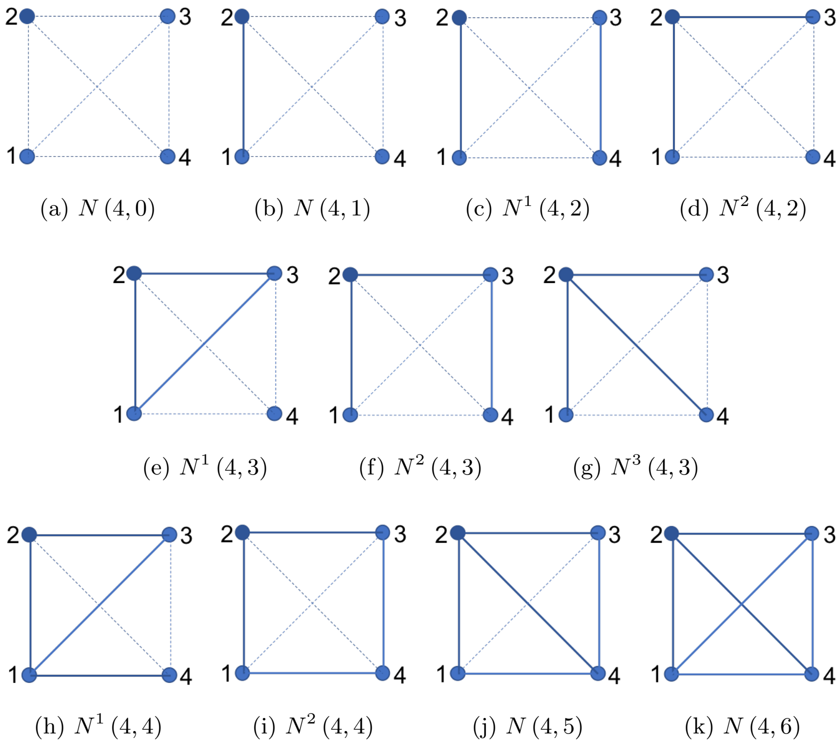
The empty and the complete networks, respectively, set the lower and the upper bounds for the chance of game continuation, the average expected repayment, and the highest expected repayment for all networks with the same number of borrowers.

Clearly, the chance of game continuation Q and the average expected repayment K of the network are increasing in the chance of project success α. We will discuss later that they are also increasing in the average centrality and average connectivity of the network. Note that Q and K may also change with the group size n, but such changes may not be monotonic because a new node can be connected to the existing members in different ways.

4.1 Tetrad Networks

In this section, we explore how the structure of the network impacts Q and K. For gaining insights, we begin with networks as small as tetrads. Although, tetrads might not fully represent larger networks, they allow for various interesting network structures such as line, star, ring, core-periphery, and complete structures (Figures 1).

Tables 1 and 2 summarize the chance of game continuation and the average expected repayment for all possible tetrad network structures. Each row is devoted to a specific structure, and columns represent the cases in which one, two, or three members default. In any tetrad, when no one defaults, the chance of the game continuation is α^4 and the average expected repayment is $\alpha^4 R$. To avoid repetition, these cases are not included in the Tables. Notations $Q_{j,e}$ and $K_{j,e}R$, respectively,



(a) $N(4,0)$ (b) $N(4,1)$ (c) $N^1(4,2)$ (d) $N^2(4,2)$

(e) $N^1(4,3)$ (f) $N^2(4,3)$ (g) $N^3(4,3)$

(h) $N^1(4,4)$ (i) $N^2(4,4)$ (j) $N(4,5)$ (k) $N(4,6)$

Figure 1: Possible tetrad network structures.

stand for the chance of game continuation and the average expected repayment for a network with j members and e strong links. Superscripts are used when there are more than one networks with j members and e strong links. The following orders can be verified in Tables 1 and 2.

First, every added link increases both the chance of game continuation and the average expected repayment for tetrads; that is,

$$Q_{4,0} < Q_{4,1} < Q^1_{4,2}, Q^2_{4,2} < Q^1_{4,3}, Q^2_{4,3}, Q^3_{4,3} < Q^1_{4,4}, Q^2_{4,4} < Q_{4,5} < Q_{4,6},$$
$$K_{4,0} < K_{4,1} < K^1_{4,2}, K^2_{4,2} < K^1_{4,3}, K^2_{4,3}, K^3_{4,3} < K^1_{4,4}, K^2_{4,4} < K_{4,5} < K_{4,6}.$$

Although, the total degree of the network can order the overall chance of game continuation and average expected repayment for tetrads, for specific cases, such an order may not be true. For example, for handling the case, in which one member defaults, the star structure (of total degree three) perform better than the ring structure (of total degree four).

Second, among the three variations of $N(4,3)$, the star structure (Figure 1g), has the highest

Table 1: The chance of game continuation when $i = 1, 2, 3$ members fails to repay.

	$i = 1$	$i = 2$	$i = 3$
$Q_{4,0}$	$\alpha^3 (1-\alpha) \left(4\beta^3\right)$	$\alpha^2 (1-\alpha)^2 \left(6\beta^4\right)$	$\alpha (1-\alpha)^3 \left(4\beta^3\right)$
$Q_{4,1}$	$\alpha^3 (1-\alpha) \left(2\beta^3 + 2\beta^2\right)$	$\alpha^2 (1-\alpha)^2 \left(4\beta^3 + 2\beta^4\right)$	$\alpha (1-\alpha)^3 \left(2\beta^2 + 2\beta^3\right)$
$Q^1_{4,2}$	$\alpha^3 (1-\alpha) \left(4\beta^2\right)$	$\alpha^2 (1-\alpha)^2 \left(2\beta^4 + 4\beta^2\right)$	$\alpha (1-\alpha)^3 \left(4\beta^2\right)$
$Q^2_{4,2}$	$\alpha^3 (1-\alpha) \left(\beta^3 + 3\beta\right)$	$\alpha^2 (1-\alpha)^2 \left(2\beta^3 + 4\beta^2\right)$	$\alpha (1-\alpha)^3 \left(2\beta^2 + \beta + \beta^3\right)$
$Q^1_{4,3}$	$\alpha^3 (1-\alpha) \left(\beta^3 + 3\beta\right)$	$\alpha^2 (1-\alpha)^2 \left(6\beta^2\right)$	$\alpha (1-\alpha)^3 \left(\beta^3 + 3\beta\right)$
$Q^2_{4,3}$	$\alpha^3 (1-\alpha) (2\beta + 2)$	$\alpha^2 (1-\alpha)^2 \left(3\beta^2 + 2\beta + 1\right)$	$\alpha (1-\alpha)^3 \left(2\beta^2 + 2\beta\right)$
$Q^3_{4,3}$	$\alpha^3 (1-\alpha) (4)$	$\alpha^2 (1-\alpha)^2 \left(3\beta^2 + 3\right)$	$\alpha (1-\alpha)^3 \left(1 + 3\beta^2\right)$
$Q^1_{4,4}$	$\alpha^3 (1-\alpha) (4)$	$\alpha^2 (1-\alpha)^2 \left(\beta^2 + 2\beta + 3\right)$	$\alpha (1-\alpha)^3 \left(\beta^2 + 2\beta + 1\right)$
$Q^2_{4,4}$	$\alpha^3 (1-\alpha) (4)$	$\alpha^2 (1-\alpha)^2 (6)$	$\alpha (1-\alpha)^3 (4\beta)$
$Q_{4,5}$	$\alpha^3 (1-\alpha) (4)$	$\alpha^2 (1-\alpha)^2 (6)$	$\alpha (1-\alpha)^3 (2 + 2\beta)$
$Q_{4,6}$	$\alpha^3 (1-\alpha) (4)$	$\alpha^2 (1-\alpha)^2 (6)$	$\alpha (1-\alpha)^3 (4)$

Table 2: The expected repayment of a successful member when $i = 1, 2, 3$ of his co-members fail to repay.

	$i = 1$	$i = 2$	$i = 3$
$K_{4,0}R$	$\alpha^3 (1-\alpha) \left(3 + \frac{6\beta}{6}\right) R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{12\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{6\beta}{2}\right) R$
$K_{4,1}R$	$\alpha^3 (1-\alpha) \left(3 + \frac{1+5\beta}{6}\right) R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{2+10\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{1+5\beta}{2}\right) R$
$K^1_{4,2}R$	$\alpha^3 (1-\alpha) \left(3 + \frac{2+4\beta}{6}\right) R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{4+8\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{2+4\beta}{2}\right) R$
$K^2_{4,2}R$	$\alpha^3 (1-\alpha) \left(3 + \frac{3+3\beta}{6}\right) R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{5+7\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{2+4\beta}{2}\right) R$
$K^1_{4,3}R$	$\alpha^3 (1-\alpha) \left(3 + \frac{3+3\beta}{6}\right) R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{6+6\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{3+3\beta}{2}\right) R$
$K^2_{4,3}R$	$\alpha^3 (1-\alpha) \left(3 + \frac{5+\beta}{6}\right) R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{8+4\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{3+3\beta}{2}\right) R$
$K^3_{4,3}R$	$\alpha^3 (1-\alpha) 4R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{9+3\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{3+3\beta}{2}\right) R$
$K^1_{4,4}R$	$\alpha^3 (1-\alpha) 4R$	$\alpha^2 (1-\alpha)^2 \left(3 + \frac{10+2\beta}{4}\right) R$	$\alpha (1-\alpha)^3 \left(1 + \frac{4+2\beta}{2}\right) R$
$K^2_{4,4}R$	$\alpha^3 (1-\alpha) 4R$	$\alpha^2 (1-\alpha)^2 6R$	$\alpha (1-\alpha)^3 \left(1 + \frac{4+2\beta}{2}\right) R$
$K_{4,5}R$	$\alpha^3 (1-\alpha) 4R$	$\alpha^2 (1-\alpha)^2 6R$	$\alpha (1-\alpha)^3 \left(1 + \frac{5+\beta}{2}\right) R$
$K_{4,6}R$	$\alpha^3 (1-\alpha) 4R$	$\alpha^2 (1-\alpha)^2 6R$	$\alpha (1-\alpha)^3 4R$

rank, the line structure (Figure 1f) has the second highest rank, and the structure that includes a ring and an isolated node (Figure 1e) has the lowest rank in terms of the overall chance of game continuation and average expected repayment; that is,

$$Q^1_{4,3} < Q^2_{4,3} < Q^3_{4,3}, \quad \text{and} \quad K^1_{4,3} < K^2_{4,3} < K^3_{4,3}.$$

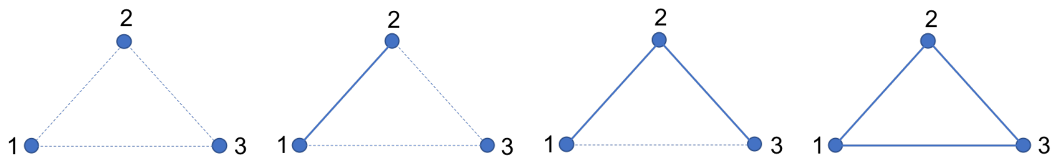
In other words, the number of second hand connections can order both the overall chance of game continuation and average expected repayment for $N(4,3)$ (a similar order holds for the two versions of $N(4,2)$ and the two versions of $N(4,4)$). However, such an order may not be true if we look at some specific cases separately. For example, in case three members default, the structure with an isolated node has a higher chance of game continuation than the line structure, while it has the same average expected repayment as the line structure; and in case two members default, the line structure perform better than the star structure in terms of a higher chance of game continuation, but it has a lower average expected repayment.

Third, the ring tetrad (Figure 1i) has a higher overall chance of game continuation and average expected repayment than the core-periphery tetrad (Figure 1h); that is,

$$Q^1_{4,4} < Q^2_{4,4}, \quad \text{and} \quad K^1_{4,4} < K^2_{4,4}.$$

The reason that the ring structure outperforms the core-periphery tetrad, although both graphs have the same total degree and the same number of strong indirect connections, may be explained by the fact that in the tetrad ring, any two nodes that are indirectly connected have two friends in common. Therefore, if any two members default, the successful members of the group can verify their truthfulness by direct or indirect monitoring and thus continue supporting them. But in the tetrad core-periphery this feature is missing. For example, if borrowers 1 and 2 default, then borrowers 3 and 4 cannot verify borrower 1's truthfulness and may stop supporting him.

However, in extreme cases, in which one or three members defaults, the core-periphery structure performs better than the ring structure because in such cases, the advantage of having more than one friends in common does not come into practice. Note that for larger ring networks such a feature may fade away, because we assume that friendship chains of a length longer than two are the same



(a) $N(3,0)$ (b) $N(3,1)$ (c) $N(3,2)$ (d) $N(3,3)$

Figure 2: Possible triad network structures.

as weak connections. Note that although, the ring structure offers a higher over all chance of game continuation and average expected repayment than the core-periphery structure, in specific cases, in which one or three members default, the core-periphery structure dominates the ring structure in terms of a higher chance of game continuation while having the same average expected repayment.

Can it benefit a tetrad group to ostracize a member? Depending on the structure, removing a borrower from a tetrad network contracts it to one of the triad networks (i.e., a network of three nodes) in Figure 2, for which the chance of game continuation and the average expected repayment are summarized in Tables 3 and 4. Columns represent the cases, in which one or two members default. In any triad, when no one defaults, the chance of the game continuation is α^3 and the average expected repayment is $\alpha^3 R$. To avoid repetition, this case is omitted from the Tables.

Claim 1. *For a tetrad network, the following statements hold.*

1. Removing a borrower of degree zero or one can increase Q and K only if α is not too small and β is not too large. Removing a borrower of degree two or three almost always decreases both Q and K.

2. Removing a borrower of degree zero increases Q for all tetrads and decreases K for all tetrads except the empty network.

3. Removing a borrower of degree one can increase Q for tetrads with lower total degree and can increase K for tetrads with a higher total degree.

4. Removing a borrower of degree two decreases both Q and K for all tetrads expect for the core-periphery.

5. Removing a borrower from a symmetrical tetrad (in which all nodes have the same degree) increases Q (except for the complete network) but decreases K.

Claim 1 suggests that removing a borrower can benefit the network when projects are not very

Table 3: Columns show the chance of game continuation when $i = 1, 2$ members fails to repay.

	$i = 1$	$i = 2$
$Q_{3,0}$	$\alpha^2 (1 - \alpha) 3\beta^2$	$\alpha (1 - \alpha)^2 3\beta^2$
$Q_{3,1}$	$\alpha^2 (1 - \alpha) (2\beta + \beta^2)$	$\alpha (1 - \alpha)^2 (2\beta + \beta^2)$
$Q_{3,2}$	$3\alpha^2 (1 - \alpha)$	$\alpha (1 - \alpha)^2 (2\beta + 1)$
$Q_{3,3}$	$3\alpha^2 (1 - \alpha)$	$3\alpha (1 - \alpha)^2$

Table 4: Columns show the average expected repayment when $i = 1, 2$ members fails to repay.

	$i = 1$	$i = 2$
$K_{3,0}R$	$\alpha^2 (1 - \alpha) (2 + \beta) R$	$\alpha (1 - \alpha)^2 (1 + 2\beta) R$
$K_{3,1}R$	$\alpha^2 (1 - \alpha) \left(2 + \frac{1+2\beta}{3}\right) R$	$\alpha (1 - \alpha)^2 \left(1 + \frac{2+4\beta}{3}\right) R$
$K_{3,2}R$	$3\alpha^2 (1 - \alpha) R$	$\alpha (1 - \alpha)^2 \left(1 + \frac{4+2\beta}{3}\right) R$
$K_{3,3}R$	$3\alpha^2 (1 - \alpha) R$	$3\alpha (1 - \alpha)^2 R$

risky or borrowers are not highly supportive of their weak links. Moreover, the conditions, under which removing a borrower can benefit the network is easier to satisfy for tetrad networks of a higher total degree. For example, the condition necessary for removing a borrower with one strong link is easier to satisfy for $N(4, 4)$ than it was for $N(4, 1)$.

Claim 1 implies that when handling risky projects, removing any borrower from a tetrad may not be a good idea even removing the borrower with the lowest degree. According to the lemma such exclusions may decrease the performance of the network both in terms of the chance of game continuation and the average expected repayment.

4.2 Polyad Networks

In this section, we try to extend the results of previous section over polyad networks with $n \geq 5$ members. First, we look for ways we can order the chance of game continuation and the average expected repayment. Lemma 1 suggests that both the number and the distribution of strong connections affect the chance of game continuation the average expected repayment of polyad networks of size n.

Lemma 1. *For any polyad network with $n \geq 5$ members, Q and K are increasing in the average degree of centrality and the average degree of connectivity of the network.*

Lemma 1 shows that a borrower can contribute to the performance of the network both by creating second-hand connections between borrowers that are not otherwise connected (captured by the degree of centrality $\overline{ce}$) and by building new second-hand connections between borrowers that are already indirectly connected (captured by the degree of connectivity $\overline{co}$). Proposition 2 employs the results of Lemma 1 to discuss when removing a borrower can improve the network performance.

Proposition 2. *Removing a borrower from a network may increase the chance of game continuation and the average expected repayment for the remaining group members only if the degrees of centrality and connectivity of the borrower relative to the average degrees of centrality and connectivity of the network are small.*

Proposition 2 implies that except when projects are highly risky, removing a borrower that has no strong tie and thus is not monitored by anyone in the group improves the performance of any network with some strong connections. In other words, connected networks, in which every member is fully monitored by at least one other member of the group, should be preferred to disconnected networks even at the cost of losing a member. According to this proposition, removing a highly connected member can never benefit a network. Such a member works like a hub connecting other group members and facilitating monitoring through out the network.

According to Proposition 2, removing an isolated node benefits different networks differently depending on how connected they are. For example, the isolated node from a pentad network that includes a ring tetrad (Figures 3c) increases the chance of game continuation and the average expected repayment for the remaining tetrad network more than removing the isolated node from the pentad network, which includes a core-periphery tetrad (Figures 3b). In other words, the better the network is connected, the more it benefits such a contraction.

Proposition 2 also implies that adding a member with a higher than average degree of centrality and connectivity may improve the network performance, especially when projects are risky. However, enlarging the group should be done more carefully because adding a node to a network can change its structure in different ways.

Proposition 2 can help us sort pentad networks with four links according to their performance and compare them with tetrads. Structures including four connected nodes and an isolated node (Figures 3b and 3c) have a higher overall performance than the one including two components

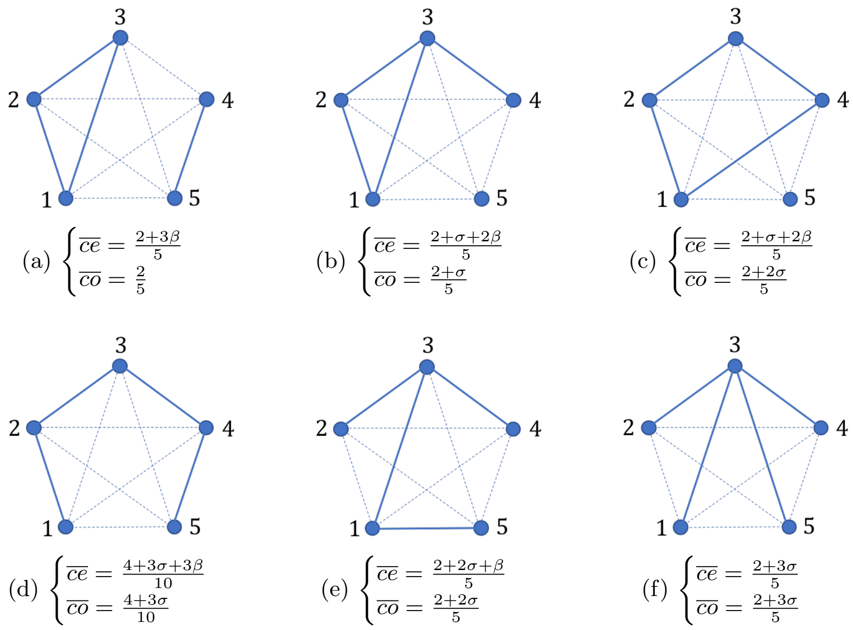


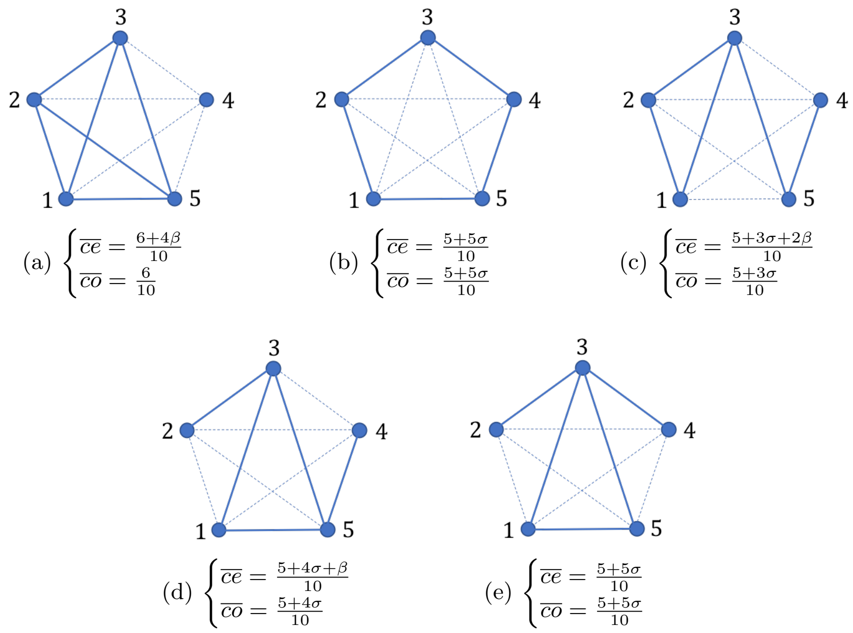
Figure 3: Pentad networks of total degree eight sorted with their average degree of centrality.

(Figure 3a) because the formers have a higher number of second-hand connections and thus a higher average degrees of centrality. Similarly, network structures which include five connected nodes (Figures 3d and 3e and 3f) have a higher overall performance than the ones with four connected nodes and an isolated node (Figures 3b and 3c).

Moreover, among the two structures with four connected nodes (Figures 3b and 3c), the one which includes a ring structure (Figure 3c) has a higher performance than the one which includes a core-periphery (Figure 3b) despite both networks have the same average degree of centrality. The reason for this is that the pentad including a tetrad ring provides some indirectly connected nodes with more than one friend in common and thus has a higher degree of connectivity than the pentad network including a tetrad core-periphery.

Further, similar to the case for tetrad networks, the star pentad (Figure 3f) has the highest overall performance among the structures of the same total degree; but, unlike the case for the tetrad networks, the ring pentad (Figure 4b) has a lower performance than all the core-periphery structures of the same total degree (Figures 4c and 4d and 4e). Although, for pentads compared to tetrads, the ring structure is less desirable, it still continues to outperform the star structure.

The above comparison between tetrads and pentads shows that the performance of some struc-



(a) $\begin{cases} \overline{ce} = \frac{6+4\beta}{10} \\ \overline{co} = \frac{6}{10} \end{cases}$
(b) $\begin{cases} \overline{ce} = \frac{5+5\sigma}{10} \\ \overline{co} = \frac{5+5\sigma}{10} \end{cases}$
(c) $\begin{cases} \overline{ce} = \frac{5+3\sigma+2\beta}{10} \\ \overline{co} = \frac{5+3\sigma}{10} \end{cases}$
(d) $\begin{cases} \overline{ce} = \frac{5+4\sigma+\beta}{10} \\ \overline{co} = \frac{5+4\sigma}{10} \end{cases}$
(e) $\begin{cases} \overline{ce} = \frac{5+5\sigma}{10} \\ \overline{co} = \frac{5+5\sigma}{10} \end{cases}$

Figure 4: Pentad networks of total degree five sorted with their average degree of centrality.

tures may be affected by the group size. Proposition 3 highlights this point for three specific network structures.

Proposition 3. *For polyad networks with $n \geq 5$ members, both the star and the ring structures have a higher overall performance than the same-sized line structure; and a star structure has a higher overall performance than the same-sized ring structure only if $\sigma > \frac{2}{n^2-5n+2}$.*

Proposition 3 shows that, similar to the case for tetrads and pentads, the star structure polyad has a higher overall performance than the line structure polyad. However, unlike the case for tetrads and pentads, the star structure polyad can have a higher overall performance than the ring structure polyad if the group is large enough and second-hand connections are valued highly by borrowers. In other words, having many second-hand links may well compensates lacking a few direct links if the second-hand connections are relatively well trusted by borrowers.

This proposition emphasizes that enlarging the group adds to the relative attraction of the star structure while reduces the relative appeal for the ring structure, so that for a larger group size, the star structure always has a higher overall performance than the ring structure. Intuitively, a star network with $n+1$ members compared to a star network with n members has one extra strong link and $n-1$ extra indirect links; while a ring network of $n+1$ members compared to a ring network with n members has one extra strong link, one extra indirect link, and n extra weak links. Thus, the star structure benefits from enlarging the group more than the ring structures do.

We recall from the previous section that some network structures, despite not having the highest overall performance, may dominate the other structures for handling specific cases of project risk. Can the line or the ring polyads proceed the star polyad for handling specific cases of project risk?

Proposition 4. *For polyad networks of $n \geq 5$ fixed members, the following statements hold.*

1. When one member defaults, a) both the ring and the star structures have a higher performance than the line structure; b) the star structure can have a higher average expected repayment than the ring structure if $\sigma - \beta$ and n are large enough and a higher chance of game continuation than the ring structure always.

2. When $n-1$ members default, a) the ring structure has a higher performance than the line structure always; b) the star structure can have a higher performance than the line structure if β is neither too large and nor too small; c) and the star structure can have a higher chance of game continuation than the ring structure if β is small enough and n is large enough and a higher average expected repayment always.

Considering that the number of failures in a group is related to the riskiness of projects, Proposition 4 implies that in extreme cases, where projects are highly risky or only slightly risky, the star structure may have an advantage over the line and the ring structures in terms of providing a higher chance of game continuation and a higher average expected repayment, especially when borrowers value their second-hand links notably higher than their weak links.

The intuition behind this result maybe explained as follows. In a borrower network of n members, when one member defaults, the line network is likely to be partitioned into two subgroups, which have only weak access to each other's information, and the ring network will be reduced to a line network, while the star network has a high chance of staying a star network. In other words, with one member defaulting on his repayment, the star network is the only network that is likely to keep its members' links intact and thus keep the average expected repayment high. Such an effect is further magnified in larger groups.

When $n-1$ borrowers fail to repay, the only successful member in a line network would be able to verify the truthfulness of either one close friend if he is one the endpoints or two close friends if he is a midpoint; while a successful borrower in a ring can always verify the truthfulness of two close friends in a similar situation. Thus, the ring network has a slightly higher chance to fulfill the mission than the line network when projects are risky. In such a situation, in a star network, although it is likely that the only successful member is one of the peripheries, who is able to verify the truthfulness of only one close friend, there is always a little chance that the successful node is the core node, who would be able to verify the truthfulness of every body and repay the total group loan. Note that in large groups, whether the only successful member can verify the truthfulness of one or two members out of $n-1$ defaulting members would not make a meaningful difference.

4.3 Optimal Group Size

We recall from Proposition 1 that the relationship between the group size and the borrower welfare depend on the structure of the network both before and after enlarging the group. In this section, we consider expanding (and contracting) a group in a way that its structure stays intact, and discuss how such expansions (and contractions) affects the group performance and consequently the borrower welfare.

In the previous section (Propositions 3 and 4), we discussed that for larger borrower groups, star structures not only have a higher overall performance than ring and line structures of the same size but also has a higher performance for handling extreme cases of project risk. Although, a larger group improves the comparative status of the star structure with respect to the ring and the line structures, it is not yet clear that whether enlarging these structures can improve their absolute performance and the welfare of their borrowers.

It is rather intuitive that for star networks, the highest expected repayment k_m^s in the network comes from the core borrower, thus clearly, k_m^s increases if the group grows larger, while for ring and line networks the highest expected repayment (k_m^r and k_m^l respectively) in the network does not change with the group size n. In order to check how enlarging the group affects the chance of game continuation Q and the average expected repayment K, we first formulate both of these factors in the following lemma.

Lemma 2. *1. For a star network of $n \geq 5$,*

$$\begin{aligned}
Q^s &= \alpha^n + \sum_{i=1}^{n-1} \alpha^{n-i} (1-\alpha)^i \left[\binom{n-1}{i} \left(1\sigma^i\right)^{n-1-i} + \binom{n-1}{i-1} \left(1\beta^{i-1}\right)^{n-i} \right], \\
K^s &= \frac{n-1}{n} \left[\alpha^n + \alpha (1-\alpha)^{n-1} (2 + (n-2)\beta) \right] \\
&\quad + \frac{n-1}{n} \sum_{i=1}^{n-2} \alpha^{n-i} (1-\alpha)^i \left[\binom{n-2}{i} \left(1 + \frac{i\sigma}{n-i}\right) + \binom{n-2}{i-1} \left(1 + \frac{1+(i-1)\beta}{n-i}\right) \right] \\
&\quad + \frac{1}{n} \sum_{i=0}^{n-1} \alpha^{n-i} (1-\alpha)^i \binom{n-1}{i} \left(1 + \frac{i}{n-i}\right).
\end{aligned}$$

2. For a ring network of size $n \geq 5$,

$$\begin{aligned}
Q^r &= Q^{r_0} + Q^{r_1} + Q^{r_2} + \cdots + Q^{r_{n-1}}, \\
K^r &= \sum_{i=0}^{n-1} \sum_{j=\max\{i-2-(n-5),0\}}^{\min\{2,i\}} \sum_{k=\max\{i-j-(n-5),0\}}^{\min\{2,(i-j)\}} \\
&\quad \left[\binom{2}{j} \binom{2}{k} \binom{n-5}{i-j-k} \alpha^{n-i} (1-\alpha)^i \left(1 + \frac{j + k\sigma + (i-j-k)\beta}{n-i}\right) \right],
\end{aligned}$$

where

$$\begin{aligned}
Q^{r_0} &= \alpha^n \\
Q^{r_1} &= n\alpha^{n-1} (1-\alpha) \left(1^2 \sigma^2 \beta^{n-5}\right), \\
&\vdots \quad \vdots \quad \vdots \\
Q^{r_{n-1}} &= n\alpha (1-\alpha)^{n-1} \left(1^2 \sigma^2 \beta^{n-5}\right),
\end{aligned}$$

and

$$P(h_i) = \begin{cases} 1 & h_i = 0, 1 \\ \sigma^2 & h_i = 2 \\ \sigma^2 \beta^{2h_i - 4} & h_i \geq 3 \end{cases}.$$

3. For a line structure network of size $n \geq 5$,

$$\begin{aligned}
Q^{l} &= Q^{l_0} + Q^{l_1} + Q^{l_2} + \cdots + Q^{l_{n-1}}, \\
K^{l} &= \frac{2}{n} \sum_{i=0}^{n-1} \sum_{j=\max\{i-1-(n-3),0\}}^{\min\{1,i\}} \sum_{k=\max\{i-j-(n-3),0\}}^{\min\{1,(i-j)\}} \\
&\left[\binom{1}{j} \binom{1}{k} \binom{n-3}{i-j-k} \alpha^{n-i} (1-\alpha)^{i} \left(1 + \frac{j + k\sigma + (i-j-k)\beta}{n-i} \right) \right] \\
&+ \frac{2}{n} \sum_{i=0}^{n-1} \sum_{j=\max\{i-1-(n-4),0\}}^{\min\{2,i\}} \sum_{k=\max\{i-j-(n-4),0\}}^{\min\{1,(i-j)\}} \\
&\left[\binom{2}{j} \binom{1}{k} \binom{n-4}{i-j-k} \alpha^{n-i} (1-\alpha)^{i} \left(1 + \frac{j + k\sigma + (i-j-k)\beta}{n-i} \right) \right] \\
&+ \frac{n-4}{n} \sum_{i=0}^{n-1} \sum_{j=\max\{i-2-(n-5),0\}}^{\min\{2,i\}} \sum_{k=\max\{i-j-(n-5),0\}}^{\min\{2,(i-j)\}} \\
&\left[\binom{2}{j} \binom{2}{k} \binom{n-5}{i-j-k} \alpha^{n-i} (1-\alpha)^{i} \left(1 + \frac{j + k\sigma + (i-j-k)\beta}{n-i} \right) \right],
\end{aligned}$$

where

$$\begin{aligned}
Q^{l_0} &= \alpha^{n}, \\
Q^{l_1} &= 2\alpha^{n-1}(1-\alpha)\left(1\sigma\beta^{n-3}\right) + 2\alpha^{n-1}(1-\alpha)\left(1^2\sigma\beta^{n-4}\right) + (n-4)\,\alpha^{n-1}(1-\alpha)\left(1^2\sigma^2\beta^{n-5}\right), \\
&\vdots \quad \vdots \quad \vdots \\
Q^{l_{n-1}} &= 2\alpha(1-\alpha)^{n-1}\left(1\sigma\beta^{n-3}\right) + 2\alpha(1-\alpha)^{n-1}\left(1^2\sigma\beta^{n-4}\right) + (n-4)\,\alpha(1-\alpha)^{n-1}\left(1^2\sigma^2\beta^{n-5}\right).
\end{aligned}$$

Using the formulas suggested by Lemma 2, Lemma 3 verifies whether the chance of game continuation and the average expected repayment can grow larger in larger group with a star, line, or ring structure.

Lemma 3. *For star, ring, and line networks, the chance of game continuation is always decreasing in n, but the average expected repayment can be increasing in n if α is small.*

Thus, enlarging a network with less than complete structure may not improve the chance of game continuation and the average expected repayment unless when projects are risky.

Lemma 4 shows that affecting the chance of game continuation, the average expected repayment, and the highest expected repayment in the network is not the only way that enlarging the group size can influence the borrower welfare.

Lemma 4. *For a borrower network, if* $\left(K - n\frac{\partial K}{\partial n}\right) Q + \left(n\frac{\partial Q}{\partial n} - \frac{\partial k_m}{\partial n}\right) K + \frac{\partial K}{\partial n} k_m > 0$*, the loan ceiling* $\mathscr{L}$ *increases in the group size* n *for any*

$$\delta \in \left(\frac{K - n\frac{\partial K}{\partial n}}{\left(K - n\frac{\partial K}{\partial n}\right) Q + \left(n\frac{\partial Q}{\partial n} - \frac{\partial k_m}{\partial n}\right) K + \left(\frac{\partial K}{\partial n}\right) k_m}, \frac{n}{nQ + n\alpha - k_m} \right); \tag{5}$$

the borrower welfare V^R *can increase in* n *only if the loan size* L *increases in* n.

Lemma 4 proves that depending on the network structure, enlarging the group may allow an increase in the loan ceiling for mid range borrower discount factors. Consequently, for such feasible discount factors, increasing the loan ceiling improves the borrower welfare. However, Proposition 5 proves that for complete networks, increasing n or α tightens the range of the feasible discount factors, characterized by (5); and thus the group size cannot grow infinitely large, especially when the chance of project success is high. This proposition also sets an upper bound on the group size for structures other than complete structure, for which the network performance can be less often improved by enlarging the group size.

Proposition 5. *For a complete network, an* $\bar{\alpha}$ *exists such that for any* $\alpha < \bar{\alpha}$*, the borrower welfare will be increasing in*

$$n \in \left[2, \min\left\{ \left\lfloor \hat{\delta}^{-1}(\delta, \alpha) \right\rfloor, \left\lfloor \tilde{\delta}^{-1}(\delta, \alpha) \right\rfloor \right\} \right], \tag{6}$$

where $\hat{\delta}(n, \alpha) = \frac{[1-(1-\alpha)^n] + (1-\alpha)^n \ln(1-\alpha)^n}{[1-(1-\alpha)^n]^2}$*, and* $\tilde{\delta}(n, \alpha) = \frac{1}{\alpha + \frac{n-1}{n}[1-(1-\alpha)^n]}$.

Proposition 5 proves first that for a complete network of borrowers, depending on the riskiness of projects, there is an optimal group size that maximizes the borrower welfare and then finds the optimal group size. Intuitively, a larger group provides a borrower with a higher repayment insurance in case he defaults on his repayment, but a larger group also demands a higher repayment responsibility in case the borrower succeeds in his projects. Thus when projects are relatively safe and the insurance effect of a large group is less needed, enlarging a group cannot improve the borrower welfare.

Table 5: The Range of Feasible Group Sizes for Complete Networks for Some Given α

	$\alpha < 0.08$	$\alpha < 0.26$	$\alpha < 0.41$	$\alpha < 0.47$	$\alpha < 0.56$	$\alpha < 0.71$
$n = 2$	✓	✓	✓	✓	✓	✓
$n = 3$	✓	✓	✓	✓	✓	
$n = 4$	✓	✓	✓	✓		
$n = 5$	✓	✓	✓			
$\vdots$	✓	$\vdots$				
$n = 10$	✓	✓				
$\vdots$	$\vdots$					
$n = 50$	✓					

Using the proof of Proposition 5, one could simply verify that for $\bar{\alpha} = 0.47$ (i.e., for any $\alpha < 0.47$), the borrower welfare increases in $n = 2, 3, 4$, while for $\bar{\alpha} = 0.71$, the only feasible group size is $n = 2$. Table 5 provides more examples of this kind and shows that larger group sizes are feasible for a smaller chance of project success. As shown in this table, if $\alpha > 0.56$, the only feasible group size will be $n = 2$. Thus, according to Proposition 5, the borrower welfare can increase in group size only if $\alpha < 0.56$ and the given δ is such that $\hat{\delta}(n, \alpha) < \delta < \tilde{\delta}(n, \alpha)$.

5 Conclusion

This paper presents a study of how the structure of borrowers social network may affect the outcome of group lending under joint liability and what characteristics a potential optimal network structure should have. Extant microfinance literature argues that borrowers social collateral may facilitate cooperation and substitute contract enforcement (e.g., Besley and Coate, 1995). Aligned with this literature, we provide evidence that stronger social connectedness can improve repayment rate. We differ from this literature by explicitly modeling social ties as means of peer monitoring. In our model, a borrower that is strongly connected to the network better monitors and is better monitored, and thus has a higher chance of both being backed by peers in case of non-strategic default and being punished by peers in case of strategic default.

We prove that the borrower welfare is maximized in a complete borrower network, in which borrowers can fully monitor each other's project output and are able to differentiate strategic defaults from non-strategic defaults. The intuition behind this result is simple. With full monitoring, borrowers pay for their non-strategic defaulting co-member and punish their strategic defaulting co-member, both of which improve cooperation. We also prove that a higher average degree of centrality and connectivity in the borrower social network increases the chance of game continuation and the average expected repayment, which in turn increases the borrower welfare. We argue that in order to maintain a relatively high average degrees of centrality and connectivity, connected groups should be preferred to disconnected ones.

We focus especially on the star and the ring structure networks, which represent highly authoritarian and highly egalitarian networks, respectively. We argue that if project risk is not excessively high, the egalitarian ring structure may proceed the authoritarian star structure of the same size both in terms of a higher chance of project success and a higher average expected repayment. While for average project risk, the performance is higher in authoritarian networks compared to the egalitarian networks, especially for larger groups.

The stylized model in this paper is limited in many ways and could be extended by loosening any of our assumptions. An immediate extension of our work is considering a longer chain of friendships and not only friends of friends. In such a case, we might end up with a combination of ring and star networks as an optimal choice, as a star network itself is a core-periphery structure with a single node as the core. One could also look at contracts that are not necessarily the same for all group members. The contract could depend on the level of social capital each individual borrower brings to the group (his degree of centrality in our case) instead of the collective social capital of the group. Such an extension is also in line with the recent shift from explicit joint liability to individual lending by some MFIs, which interestingly preserve some of the group lending traditions. Another interesting extension of our research is to look at strategies less severe than grim-trigger. The grim-trigger strategy in our setting assumes that borrowers punish free-riding in their own cost. In the real world, it may well benefit the repaying borrowers to repay for their strategically defaulting group members and proceed to the next round of the lending game.

The setting under study in this paper, apart from the microlending, can emerge in other social dilemma contexts that deal with monitoring, where the interests of the individual and of the group are not aligned and agents exert externalities on one another.

References

Acemoglu, D., A. Ozdaglar, and A. Tahbaz-Salehi (2016), Networks, shocks, and systemic risk, in *The Oxford Handbook of the Economics of Networks*, Oxford University Press.

Ahlin, C. (2015), The role of group size in group lending, *Journal of Development Economics*, *115*, 140–155.

Ambrus, A., M. Mobius, and A. Szeidl (2014), Consumption risk-sharing in social networks, *The American Economic Review*, *104*(1), 149–182.

Armendáriz, B., and J. Morduch (2010), *The Economics of Microfinance*, MIT press.

Attanasio, O., A. Barr, J. C. Cardenas, G. Genicot, and C. Meghir (2012), Risk pooling, risk preferences, and social network, *American Economic Journal: Applied Economics*, *4*(2), 134–167.

Baker, W. E., and R. R. Faulkner (1993), The social organization of conspiracy: Illegal networks in the heavy electrical equipment industry, *American Sociological Review*, pp. 837–860.

Bala, V., and S. Goyal (2000), A noncooperative model of network formation, *Econometrica*, *68*(5), 1181–1229.

Besley, T., and S. Coate (1995), Group lending, repayment incentives and social collateral, *Journal of Development Economics*, *46*(1), 1–18.

Borgatti, S. P., and D. S. Halgin (2011), On network theory, *Organization Science*, *22*(5), 1168–1181.

Bramoullé, Y., and R. Kranton (2007b), Risk sharing across communities, *The American Economic Review*, *97*(2), 70–74.

Canales, R., and J. Greenberg (2015), A matter of (relational) style: loan officer consistency and exchange continuity in microfinance, *Management Science*, *62*(4), 1202–1224.

Carpenter, J., S. Kariv, and A. Schotter (2012), Network architecture, cooperation and punishment in public good experiments, *Review of Economic Design*, *16*(2-3), 93–118.

Carpenter, J. P. (2007), Punishing free-riders: How group size affects mutual monitoring and the provision of public goods, *Games and Economic Behavior*, *60*(1), 31–51.

Carson, J. B., P. E. Tesluk, and J. A. Marrone (2007), Shared leadership in teams: An investigation of antecedent conditions and performance, *Academy of Management Journal*, *50*(5), 1217–1234.

Cason, T. N., L. Gangadharan, and P. Maitra (2012), Moral hazard and peer monitoring in a laboratory microfinance experiment, *Journal of Economic Behavior & Organization*, *82*(1), 192–209.

Cassar, A., L. Crowley, and B. Wydick (2007), The effect of social capital on group loan repayment: Evidence from field experiments, *The Economic Journal*, *117*(517), F85–F106.

Collins, C. J., and K. D. Clark (2003), Strategic human resource practices, top management team social networks, and firm performance: The role of human resource practices in creating organizational competitive advantage, *Academy of Management Journal*, *46*(6), 740–751.

Cummings, J. N., and R. Cross (2003), Structural properties of work groups and their consequences for performance, *Social Networks*, *25*(3), 197–210.

De Quidt, J., T. Fetzer, and M. Ghatak (2018), Commercialization and the decline of joint liability microcredit, *Journal of Development Economics*, *134*, 209–225.

Falk, A., and M. Kosfeld (2012), It's all about connections: Evidence on network formation, *Review of Network Economics*, *11*(3).

Fehr, E., and S. Gächter (2000), Cooperation and punishment in public goods experiments, *American Economic Review*, *90*(4), 980—994.

Feigenberg, B., E. Field, and R. Pande (2013), The economic returns to social interaction: Experimental evidence from microfinance, *The Review of Economic Studies*, p. rdt016.

Ghatak, M., and T. W. Guinnane (1999), The economics of lending with joint liability: theory and practice, *Journal of Development Economics*, *60*(1), 195–228.

Gloor, P. A., M. Paasivaara, D. Schoder, and P. Willems (2008), Finding collaborative innovation networks through correlating performance with social network structure, *International Journal of Production Research*, *46*(5), 1357–1371.

Goeree, J. K., A. Riedl, and A. Ule (2009), In search of stars: Network formation among heterogeneous agents, *Games and Economic Behavior*, *67*(2), 445–466.

Grund, T. U. (2012), Network structure and team performance: The case of english premier league soccer teams, *Social Networks*, *34*(4), 682–690.

Haldar, A., and J. E. Stiglitz (2016), Group lending, joint liability, and social capital: Insights from the indian microfinance crisis, *Politics & Society*, *44*(4), 459–497.

Hermes, N., R. Lensink, and H. T. Mehrteab (2005), Peer monitoring, social ties and moral hazard in group lending programs: Evidence from eritrea, *World Development*, *33*(1), 149–169.

Inkpen, A. C., and E. W. Tsang (2005), Social capital, networks, and knowledge transfer, *Academy of Management Review*, *30*(1), 146–165.

Kalnins, A., and W. Chung (2006), Social capital, geography, and survival: Gujarati immigrant entrepreneurs in the us lodging industry, *Management Science*, *52*(2), 233–247.

Karlan, D., M. Mobius, T. Rosenblat, and A. Szeidl (2009), Trust and social collateral, *The Quarterly Journal of Economics*, *124*(3), 1307–1361.

Kumar, R., J. Novak, and A. Tomkins (2010), Structure and evolution of online social networks, in *Link Mining: Models, Algorithms, and Applications*, pp. 337–357, Springer.

Leibbrandt, A., A. Ramalingam, L. Sääksvuori, and J. M. Walker (2015), Incomplete punishment networks in public goods games: experimental evidence, *Experimental Economics*, *18*(1), 15–37.

Leider, S., M. M. Möbius, T. Rosenblat, and Q.-A. Do (2009), Directed altruism and enforced reciprocity in social networks, *The Quarterly Journal of Economics*, *124*(4), 1815–1851.

Lin, M., N. R. Prabhala, and S. Viswanathan (2013), Judging borrowers by the company they keep: Friendship networks and information asymmetry in online peer-to-peer lending, *Management Science*, *59*(1), 17–35.

Mobius, M., and T. Rosenblat (2016), Informal transfers in social networks, in *The Oxford Handbook of the Economics of Networks*, Oxford University Press.

Paal, B., and T. Wiseman (2011), Group insurance and lending with endogenous social collateral, *Journal of Development Economics*, *94*(1), 30–40.

Reagans, R., and E. W. Zuckerman (2001), Networks, diversity, and productivity: The social capital of corporate r&d teams, *Organization Science*, *12*(4), 502–517.

Rezaei, B., S. Dasu, and R. Ahmadi (2017), Optimal group size in joint liability contracts, *Decision Analysis*, *14*(3), 204–225.

Rulke, D. L., and J. Galaskiewicz (2000), Distribution of knowledge, group network structure, and group performance, *Management Science*, *46*(5), 612–625.

Sparrowe, R. T., R. C. Liden, S. J. Wayne, and M. L. Kraimer (2001), Social networks and the performance of individuals and groups, *Academy of Management Journal*, *44*(2), 316–325.

Wydick, B. (1999), Can social cohesion be harnessed to repair market failures? evidence from group lending in guatemala, *Economic Journal*, pp. 463–475.

Zeller, M. (1998), Determinants of repayment performance in credit groups: The role of program design, intragroup risk pooling, and social cohesion, *Economic Development and Cultural Change*, *46*(3), 599–620.

Appendix

Proof of Proposition 1. Assuming that borrower $j = m$ is the borrower with the highest expected repayment in the network, the lender's optimization problem can be written as

$$\begin{aligned} \underset{L,R}{maximize} \quad V^R &= \frac{n\alpha Y^H - nKR}{1-\delta Q} \\ s.t. \qquad & R \leq \frac{Y^H}{n} \\ & R \leq \frac{\delta v_m^R}{n} \\ & R \geq \frac{L(1+\epsilon)}{K}. \end{aligned}$$

Since V^R is decreasing in R, the lender sets the lowest R possible; that is, $R = \frac{L(1+\epsilon)}{K}$. Replacing R into the objective function and constraints, we will have

$$\underset{L,R}{maximize} \quad V^R = \frac{n\alpha Y^H - nL(1+\epsilon)}{1-\delta Q} \tag{7}$$

$$s.t. \quad L \leq \frac{KY^H}{n(1+\epsilon)} \equiv \tilde{L}\left(K, Y^H, n, \epsilon\right) \tag{8}$$

$$L \leq \frac{\alpha\delta KY^H}{(n - n\delta Q + \delta k_m)(1+\epsilon)} \equiv \hat{L}\left(K, Q, Y^H, n, \epsilon, \alpha, \delta\right) \tag{9}$$

Constraints (8) or (9) determine the loan ceiling, that is, a feasible loan L must satisfy $L \leq \min\left\{\tilde{L}, \hat{L}\right\}$. It is not difficult to verify that $\tilde{L} \leq \hat{L}$ if and only if

$$\frac{1}{n} \leq \frac{\alpha\delta}{n - n\delta Q + \delta k_m},$$

or

$$\delta \geq \frac{n}{nQ + n\alpha - k_m} \equiv \tilde{\delta}(K, Q, n, \alpha).$$

Thus, $L \leq \tilde{L}$ if $\delta \geq \tilde{\delta}$, and (8) determines the loan ceiling; and $L \leq \hat{L}$ if $\delta \leq \tilde{\delta}$, and (9) determines the loan ceiling.□

Proof of Lemma 1. First, assume q_j defines the chance of game continuation if borrower j defaults; and $k_i R$ defines the expected repayment of borrower j when he repays. Then the chance of game continuation and the average expected repayment of the network can be written as follows.

$$
\begin{aligned}
Q &= \alpha^n + \frac{\sum_{j=1}^{n} q_j}{n}, \\
KR &= \frac{\sum_{j=1}^{n} k_j R}{n}.
\end{aligned}
$$

Second, in our model, if borrower j defaults along with another $i-1$ members of the group, a successful strongly connected friend of j would pay $\frac{R}{n-i}$ for him, a successful strongly connected friend of friend of j would pay $\frac{\sigma R}{n-i}$ for him if the friend in common is repaying, and a successful weakly connected friend of j would pay $\frac{\beta R}{n-i}$ for him, where $\beta < \sigma < 1$. In turn, if one of the connections of borrower j defaults along with other $i-1$ members of the group, borrower j would pay $\frac{\gamma R}{n-i}$ for him if he is a strongly connected friend, would pay $\frac{\sigma\gamma R}{n-i}$ for him if he is a friend of friend and if their friend in common is repaying, and would pay $\frac{\beta\gamma R}{n-i}$ for him if he is a weakly connected friend. Moreover, a friend of friend of borrower j can add to his chance of game continuation and expected repayment only if the friend in common does not default on his repayment, thus such connections would come into practice more often if they are backed by more than one friend in common. Thus, a well-connected borrower is more likely to be supported by the network when he defaults and is more likely to support defaulting peers when he repays. More precisely, the larger the degrees of centrality and connectivity of a borrower j is, the larger q_j and k_j will be. Consequently, the larger the average degrees of centrality and connectivity of a network is, the larger Q and K will be. □

Proof of Proposition 2. We know from Lemma 1 that apart from α, the average degrees of centrality and connectivity of a network affect its overall performance (captured by Q and K) positively. Thus, if a node has a low degrees of centrality and connectivity (lower than the average of the network), he receives little (lower than average) support from the successful group members when he defaults and gives little (lower than average) support to the defaulting members when he succeeds. In other words, his defaults decreases the average chance of game continuation of the group while his success does not increase the average expected repayment. Therefore, the network is better off without such a member. □

Proof of Proposition 3. For a polyad with a line structure, the degree of centrality of nodes 1 and n is $\frac{1+\sigma+(n-3)\beta}{n-1}$ and of nodes 2 and $n-1$ is $\frac{2+\sigma+(n-4)\beta}{n-1}$ and of other $n-4$ nodes is $\frac{2+2\sigma+(n-5)\beta}{n-1}$. For such a network, the degree of connectivity of nodes 1 and n is $\frac{1+\sigma}{n-1}$ and of the nodes 2 and $n-1$ is $\frac{2+\sigma}{n-1}$ and of the other $n-4$ nodes is $\frac{2+2\sigma}{n-1}$. Thus, for a line polyad, the average degree of centrality and the average degree of connectivity, respectively, will be

$$\begin{aligned}
\overline{ce}^l &= \frac{2\left(\frac{1+\sigma+(n-3)\beta}{n-1}\right)+2\left(\frac{2+\sigma+(n-4)\beta}{n-1}\right)+(n-4)\left(\frac{2+2\sigma+(n-5)\beta}{n-1}\right)}{n} \\
&= \frac{2(n-1)+2(n-2)\sigma+\left(n^2-5n+6\right)\beta}{n(n-1)}, \\
\overline{co}^l &= \frac{2\left(\frac{1+\sigma}{n-1}\right)+2\left(\frac{2+\sigma}{n-1}\right)+(n-4)\left(\frac{2+2\sigma}{n-1}\right)}{n} \\
& \frac{2(n-1)+2(n-2)\sigma}{n(n-1)}.
\end{aligned}$$

For a polyad with a star structure, both the degree of centrality and the degree of connectivity of each periphery node are $\frac{1+(n-2)\sigma}{n-1}$ and of the core node are $\frac{n-1}{n-1}$. Thus, for a star polyad, the average degree of centrality and the average degree of connectivity, respectively, will be

$$\begin{aligned}
\overline{ce}^s = \overline{co}^s &= \frac{(n-1)\left(\frac{1+(n-2)\sigma}{n-1}\right)+\left(\frac{n-1}{n-1}\right)}{n} \\
&= \frac{2+(n-2)\sigma}{n},
\end{aligned}$$

For a polyad with a ring structure, the degree of centrality of each node is $\frac{2+2\sigma+(n-5)\beta}{n-1}$, which equals the average degree of centrality of the network $\overline{ce}^r$. For such a network, the degree of connectivity of each node is $\frac{2+2\sigma}{n-1}$, which again equals the average degree of connectivity of the network $\overline{co}^r$. That is,

$$\begin{aligned}
\overline{ce}^r &= \frac{2+2\sigma+(n-5)\beta}{n-1}, \\
\overline{co}^r &= \frac{2+2\sigma}{n-1}.
\end{aligned}$$

Knowing all the degrees of centrality and connectivity, we verify that the star structure has a higher overall performance than the line structure of the same size.

$$\begin{aligned}
\overline{ce}^l < \overline{ce}^s &\Leftrightarrow \frac{2(n-1)+2(n-2)\sigma+\left(n^2-5n+6\right)\beta}{n(n-1)} < \frac{2+(n-2)\sigma}{n} \\
&\Leftrightarrow \beta < \sigma, \\
\overline{co}^l < \overline{co}^s &\Leftrightarrow \frac{2(n-1)+2(n-2)\sigma}{n(n-1)} < \frac{2+(n-2)\sigma}{n} \\
&\Leftrightarrow 1 < n,
\end{aligned}$$

both of which hold always.

The ring structure can have a higher overall performance than the same-sized star structure only in smaller groups.

$$\begin{aligned}
\overline{ce}^s < \overline{ce}^r &\Leftrightarrow \frac{2+(n-2)\sigma}{n} < \frac{2+2\sigma+(n-5)\beta}{n-1} \\
&\Leftrightarrow \sigma < \frac{\left(n^2-5n\right)\beta+2}{n^2-5n+2}, \\
\overline{co}^s < \overline{co}^r &\Leftrightarrow \frac{2+(n-2)\sigma}{n} < \frac{2+2\sigma}{n-1} \\
&\Leftrightarrow \sigma < \frac{2}{n^2-5n+2},
\end{aligned}$$

both of which hold if $\sigma < \frac{2}{n^2-5n+2}$.

The ring structure has a higher overall performance than the same-sized line structure.

$$\begin{aligned}
\overline{ce}^l < \overline{ce}^r &\Leftrightarrow \frac{2(n-1)+2(n-2)\sigma+\left(n^2-5n+6\right)\beta}{n(n-1)} < \frac{2+2\sigma+(n-5)\beta}{n-1} \\
&\Leftrightarrow \beta < \frac{1+2\sigma}{3}, \\
\overline{co}^l < \overline{co}^r &\Leftrightarrow \frac{2(n-1)+2(n-2)\sigma}{n(n-1)} < \frac{2+2\sigma}{n-1} \\
&\Leftrightarrow 1 < n,
\end{aligned}$$

both of which hold always. □

Proof of Proposition 4. When one member fails to repay, the chance of game continuation and the expected repayment for the line, the star, and the ring networks, respectively, are

$$\begin{aligned}
Q_1^l &= 2\alpha^{n-1}(1-\alpha)\left(1\sigma\beta^{n-3}\right)+2\alpha^{n-1}(1-\alpha)\left(1^2\sigma\beta^{n-4}\right)\\
&\quad +(n-4)\alpha^{n-1}(1-\alpha)\left(1^2\sigma^2\beta^{n-5}\right),\\
Q_1^r &= n\alpha^{n-1}(1-\alpha)\left(1^2\sigma^2\beta^{n-5}\right),\\
Q_1^s &= (n-1)\alpha^{n-1}(1-\alpha)\left(1\sigma^{n-2}\right)+\alpha^{n-1}(1-\alpha)\left(1^{n-1}\right),
\end{aligned}$$

and

$$\begin{aligned}
K_1^l &= \frac{2}{n}\alpha^{n-1}(1-\alpha)\left(1+\frac{1}{n-1}+1+\frac{\sigma}{n-1}+(n-3)+\frac{(n-3)\beta}{n-1}\right)\\
&\quad +\frac{2}{n}\alpha^{n-1}(1-\alpha)\left(2+\frac{2}{n-1}+1+\frac{\sigma}{n-1}+(n-4)+\frac{(n-4)\beta}{n-1}\right)\\
&\quad +\frac{n-4}{n}\alpha^{n-1}(1-\alpha)\left(2+\frac{2}{n-1}+2+\frac{2\sigma}{n-1}+(n-5)+\frac{(n-5)\beta}{n-1}\right),\\
K_1^r &= \alpha^{n-1}(1-\alpha)\left(2+\frac{2}{n-1}+2+\frac{2\sigma}{n-1}+(n-5)+\frac{(n-5)\beta}{n-1}\right),\\
K_1^s &= \frac{n-1}{n}\alpha^{n-1}(1-\alpha)\left((n-2)+\frac{(n-2)\sigma}{n-1}+1+\frac{1}{n-1}\right)\\
&\quad +\frac{1}{n}\alpha^{n-1}(1-\alpha)(1+(n-1)(1)),
\end{aligned}$$

or more simplified

$$\begin{aligned}
K_1^l &= \alpha^{n-1}(1-\alpha)\left((n-1)+\frac{(2n-2)+(2n-4)\sigma+\left(n^2-5n+6\right)\beta}{n(n-1)}\right),\\
K_1^r &= \alpha^{n-1}(1-\alpha)\left((n-1)+\frac{2+2\sigma+(n-5)\beta}{n-1}\right),\\
K_1^s &= \alpha^{n-1}(1-\alpha)\left((n-1)+\frac{n-1}{n}\left(\frac{(n-2)\sigma+1}{n-1}\right)+\frac{1}{n}\right).
\end{aligned}$$

When one member fails to repay, the ring can have a higher performance than the line if and only if

$$
\begin{aligned}
Q_1^l < Q_1^r &\Leftrightarrow 2\sigma\beta^{n-3} + 2\sigma\beta^{n-4} + (n-4)\,\sigma^2\beta^{n-5} < n\sigma^2\beta^{n-5} \\
&\Leftrightarrow \beta^2 + \beta < 2\sigma, \\
K_1^l < K_1^r &\Leftrightarrow \frac{(2n-2) + (2n-4)\,\sigma + \left(n^2 - 5n + 6\right)\beta}{n\,(n-1)} < \frac{2 + 2\sigma + (n-5)\,\beta}{n-1} \\
&\Leftrightarrow 6\beta < 4\sigma + 2,
\end{aligned}
$$

and the star has a higher performance than the line if and only if

$$
\begin{aligned}
Q_1^l < Q_1^s &\Leftrightarrow 2\sigma\beta^{n-3} + 2\sigma\beta^{n-4} + (n-4)\,\sigma^2\beta^{n-5} < (n-1)\,\sigma^{n-2} + 1 \\
&\Leftrightarrow 2\left(\beta^{n-3} + \beta^{n-4}\right)\sigma + (n-4)\,\beta^{n-5}\sigma^2 - (n-1)\,\sigma^{n-2} < 1, \\
K_1^l < K_1^s &\Leftrightarrow \frac{(2n-2) + (2n-4)\,\sigma + \left(n^2 - 5n + 6\right)\beta}{n\,(n-1)} < \frac{n-1}{n}\left(\frac{(n-2)\,\sigma + 1}{n-1}\right) + \frac{1}{n} \\
&\Leftrightarrow \beta < \sigma,
\end{aligned}
$$

all of which hold always.

When one member fails to repay, the star can have a higher performance than the ring if and only if

$$
\begin{aligned}
Q_1^r < Q_1^s &\Leftrightarrow n\sigma^2\beta^{n-5} < (n-1)\,\sigma^{n-2} + 1, \\
K_1^r < K_1^s &\Leftrightarrow \frac{2 + 2\sigma + (n-5)\,\beta}{(n-1)} < \frac{n-1}{n}\left(\frac{1 + (n-2)\,\sigma}{n-1}\right) + \frac{1}{n} \\
&\Leftrightarrow \left(n^2 - 5n + 2\right)\sigma - \left(n^2 - 5n\right)\beta > 2 \\
&\Leftrightarrow 2 - 2\sigma < \left(n^2 - 5n\right)(\sigma - \beta)\,,
\end{aligned}
$$

in which $Q_1^r < Q_1^s$ holds always, and $K_1^r < K_1^s$ can hold only if $\sigma - \beta$ and n are large enough (note that the restriction becomes easier to satisfy as n grows larger).

Accordingly, when $n-1$ members fails to repay, the chance of a successful group repayment and the expected repayment for the line, the ring, and the star polyads, respectively, are

$$\begin{aligned} Q^l_{n-1} &= 2\alpha\left(1-\alpha\right)^{n-1}\left(1\beta^{n-2}\right)+\left(n-2\right)\alpha\left(1-\alpha\right)^{n-1}\left(1^2\beta^{n-3}\right), \\ Q^r_{n-1} &= n\alpha\left(1-\alpha\right)^{n-1}\left(1^2\beta^{n-3}\right), \\ Q^s_{n-1} &= \left(n-1\right)\alpha^{n-1}\left(1-\alpha\right)\left(1\beta^{n-2}\right)+\alpha^{n-1}\left(1-\alpha\right)\left(1^{n-1}\right), \end{aligned}$$

and

$$\begin{aligned} K^l_{n-1} &= \frac{2}{n}\alpha\left(1-\alpha\right)^{n-1}\left(1+1+\left(n-2\right)\beta\right)+\frac{n-2}{n}\alpha\left(1-\alpha\right)^{n-1}\left(1+2+\left(n-3\right)\beta\right), \\ K^r_{n-1} &= \alpha\left(1-\alpha\right)^{n-1}\left(1+2+\left(n-3\right)\beta\right), \\ K^s_{n-1} &= \frac{n-1}{n}\alpha\left(1-\alpha\right)^{n-1}\left(1+1+\left(n-2\right)\beta\right)+\frac{1}{n}\alpha\left(1-\alpha\right)^{n-1}\left(1+\left(n-1\right)\right). \end{aligned}$$

When $n-1$ members fail to repay, the ring has a higher performance than the line if and only if

$$\begin{aligned} Q^l_{n-1} < Q^r_{n-1} &\Leftrightarrow 2\beta^{n-2}+\left(n-2\right)\beta^{n-3} < n\beta^{n-3} \\ &\Leftrightarrow \beta < 1, \\ K^l_{n-1} < K^r_{n-1} &\Leftrightarrow \frac{2}{n}\left(2+\left(n-2\right)\beta\right)+\frac{n-2}{n}\left(3+\left(n-3\right)\beta\right) < 3+\left(n-3\right)\beta \\ &\Leftrightarrow \beta < 1, \end{aligned}$$

both of which hold always; and the star has a higher performance than the line if and only if

$$\begin{aligned} Q^l_{n-1} < Q^s_{n-1} &\Leftrightarrow 2\beta^{n-2}+\left(n-2\right)\beta^{n-3} < \left(n-1\right)\beta^{n-2}+1 \\ &\Leftrightarrow \left(n-2\right)\beta^{n-3}-\left(n-3\right)\beta^{n-2} < 1, \\ K^l_{n-1} < K^s_{n-1} &\Leftrightarrow \frac{2}{n}\left(2+\left(n-2\right)\beta\right)+\frac{n-2}{n}\left(3+\left(n-3\right)\beta\right) < \frac{n-1}{n}\left(2+\left(n-2\right)\beta\right)+\frac{1}{n}\left(n\right) \\ &\Leftrightarrow \left(n-1\right) < 2\left(n-2\right)\left(n-3\right)\beta, \end{aligned}$$

in which $Q^l_{n-1} < Q^s_{n-1}$ can hold if β is not too large and $K^l_{n-1} < K^s_{n-1}$ can hold if β is not too small. Note that the restrictions are easy to satisfy and they become even easier to satisfy as n grows larger.

When $n-1$ members fail, the star can have a higher performance than the ring if and only if

$$\begin{aligned}
Q^r_{n-1} < Q^s_{n-1} &\Leftrightarrow n\beta^{n-3} < (n-1)\,\beta^{n-2} + 1,\\
K^r_{n-1} < K^s_{n-1} &\Leftrightarrow 3 + (n-3)\,\beta < \frac{n-1}{n}\left(2 + (n-2)\,\beta\right) + \frac{1}{n}\left(1 + (n-1)\right)\\
&\Leftrightarrow 1 < \beta.
\end{aligned}$$

Clearly, $Q^r_{n-1} < Q^s_{n-1}$ holds only if β is small enough or n is large enough (the restriction becomes easier to satisfy as n grows larger); while $K^r_{n-1} < K^s_{n-1}$ holds always. $\square$

Proof of Lemma 2. **Part 1.** In a star polyad, if a member fails to repay, the chance that his peers pay for him depends on whether he is the core member or one of the $n-1$ periphery members. Thus, the chance of game continuation for a star polyad can be written as follows.

$$\begin{aligned}
Q^s &= \alpha^n + \binom{n-1}{1}\alpha^{n-1}(1-\alpha)\left(\sigma^1\right)^{n-2} + \alpha^{n-1}(1-\alpha)(1)^{n-1}\\
&\quad + \binom{n-1}{2}\alpha^{n-2}(1-\alpha)^2\left(\sigma^2\right)^{n-3} + \binom{n-1}{1}\alpha^{n-2}(1-\alpha)^2\left(\beta^1\right)^{n-2}\\
&\quad \vdots\\
&\quad + \binom{n-1}{n-1}\alpha^1(1-\alpha)^{n-1}\left(\sigma^{n-1}\right)^0 + \binom{n-1}{n-2}\alpha^1(1-\alpha)^{n-1}\left(\beta^{n-2}\right)^1\\
&= \alpha^n + \sum_{i=1}^{n-1}\alpha^{n-i}(1-\alpha)^i\left[\binom{n-1}{i}\left(\sigma^i\right)^{n-1-i} + \binom{n-1}{i-1}\left(\beta^{i-1}\right)^{n-i}\right].
\end{aligned}$$

In a star network, the expected repayment varies between the core member and periphery members. Thus, the average expected repayment can be written as

$$K^s = \frac{n-1}{n}K^{s_p} + \frac{1}{n}K^{s_c},$$

in which K^{s_p} and K^{s_c}, respectively, are the expected repayment of each repaying periphery borrower and the one of the repaying core borrower and are calculated as follows,

$$\begin{aligned} K^{s_p} &= \alpha^n + \alpha^{n-1}(1-\alpha)\left[\binom{n-2}{1}\left(1+\frac{\sigma}{n-1}\right)+\binom{n-2}{0}\left(1+\frac{1}{n-1}\right)\right] \\ &\quad +\alpha^{n-2}(1-\alpha)^2\left[\binom{n-2}{2}\left(1+\frac{2\sigma}{n-2}\right)+\binom{n-2}{1}\left(1+\frac{1+\beta}{n-2}\right)\right] \\ &\quad \vdots \\ &\quad +\alpha^2(1-\alpha)^{n-2}\left[\binom{n-2}{n-2}\left(1+\frac{(n-2)\sigma}{2}\right)+\binom{n-2}{n-3}\left(1+\frac{1+(n-3)\beta}{2}\right)\right] \\ &\quad +\alpha(1-\alpha)^{n-1}\left[1+1+(n-2)\beta\right] \\ &= \alpha^n + \sum_{i=1}^{n-2}\alpha^{n-i}(1-\alpha)^i\left[\binom{n-2}{i}\left(1+\frac{i\sigma}{n-i}\right)+\binom{n-2}{i-1}\left(1+\frac{1+(i-1)\beta}{n-i}\right)\right] \\ &\quad +\alpha(1-\alpha)^{n-1}\left[2+(n-2)\beta\right]; \end{aligned}$$

and

$$\begin{aligned} K^{s_c} &= \alpha^n + \binom{n-1}{1}\alpha^{n-1}(1-\alpha)\left(1+\frac{1}{n-1}\right)+\cdots+\binom{n-1}{n-1}\alpha(1-\alpha)^{n-1}(1+(n-1)) \\ &= \sum_{i=0}^{n-1}\alpha^{n-i}(1-\alpha)^i\binom{n-1}{i}\left(1+\frac{i}{n-i}\right). \end{aligned}$$

Part 2. For a ring polyad, the chance of game continuation can be written as

$$Q^r = Q^{r_0}+Q^{r_1}+Q^{r_2}+\cdots+Q^{r_{n-1}},$$

in which Q^{r_i} is the probability of a successful group repayment when $i=0,1,...,n-1$ members fail to repay. In such a network, any member who fails to repay has at most two friends who pay for him for sure, at most two friends of friends who pay for him with probability σ, and the rest of the group who pay for him with probability β.

When no one fails to repay, the probability of a successful group repayment is $Q^{r_0}=\alpha^n$.

When only one member fails to repay, the probability of a successful group repayment will be $Q^{r_1}=n\alpha^{n-1}(1-\alpha)\left(1^2\sigma^2\beta^{n-5}\right)$.

When two members fail, the network will be divided into two sections that consist of h_1 and

$h_2 = n - 2 - h_1$ repaying members. In such a case, each of the two sections repays with probability

$$P(h_i) = \begin{cases} 1 & h_i = 0 \\ 1 & h_i = 1 \\ \sigma\sigma & h_i = 2 \\ \beta\sigma^2\beta & h_i = 3 \\ \beta(\sigma\beta)\left(\beta^2\right)^{h_i-4}(\sigma\beta)\beta & h_i \geq 4 \end{cases},$$

or more simplified

$$P(h_i) = \begin{cases} 1 & h_i = 0, 1 \\ \sigma^2 & h_i = 2 \\ \sigma^2\beta^{2h_i-4} & h_i \geq 3 \end{cases}.$$

Note that if one of the h_1 or $h_2 = n-2-h_1$ is zero, it means that two consecutive nodes have failed, and the graph consists of only one repaying section. Thus, when two members fail, the probability of a successful group repayment will be

$$Q^{r_2} = \sum_{h_1=0}^{n-2} n\alpha^{n-2}(1-\alpha)^2 P(h_1) P(n-2-h_1).$$

When three members fail, the probability of a successful group repayment will be

$$Q^{r_3} = \sum_{h_1=0}^{n-3}\sum_{h_2=0}^{n-3-h_1} n\alpha^{n-3}(1-\alpha)^3 P(h_1) P(h_2) P(n-3-h_1-h_2),$$

and so on for any $2 < i < n-2$.

When $n-2$ members fail, the two successful members may be direct friends, indirect friends, or just weakly connected, thus the probability of a successful group repayment will be

$$\begin{aligned} Q^{r_{n-2}} = {} & n\alpha^2(1-\alpha)^{n-2}\left(1\sigma\beta^{n-4}\right)^2 + n\alpha^2(1-\alpha)^{n-2}\left(1^2\sigma\beta^{n-5}\right)^2 \\ & + \left[\binom{n}{2} - 2n\right]\alpha^2(1-\alpha)^{n-2}\left(1^2\sigma^2\beta^{n-6}\right)^2. \end{aligned}$$

When $n-1$ members fail, the only successful member will pay with probability

$$Q^{r_{n-1}} = n\alpha\left(1-\alpha\right)^{n-1}\left(1^2\sigma^2\beta^{n-5}\right).$$

In turn, the average expected repayment for a ring polyad equals the expected repayment of one borrower, as borrowers are in symmetrical situations. When i members fail, a successful repaying borrower can have at most $j=2$ friends and $k=2$ friends of friends among them, for whom he pays $\frac{j}{n-i}$ and $\frac{k\sigma}{n-i}$. The rest of the $(i-j-k)$ defaulting members are only weak connection of the successful repaying borrower, for whom he pays $\frac{(i-j-k)\beta}{n-i}$. Thus, the expected repayment of any repaying borrower should sum up to

$$K^r = \sum_i\sum_j\sum_k \binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}\left(1-\alpha\right)^i\left(1+\frac{j+k\sigma+\left(i-j-k\right)\beta}{n-i}\right).$$

If $0\leq i\leq n-5$, a repaying borrower may have no friends or friends of friends, who failed, thus, the expected repayment should be rewritten as

$$\sum_{i=0}^{n-5}\sum_{j=0}^{\min\{2,i\}}\sum_{k=0}^{\min\{2,(i-j)\}}\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}\left(1-\alpha\right)^i\left(1+\frac{j+k\sigma+\left(i-j-k\right)\beta}{n-i}\right).$$

If $i>n-5$, a repaying borrower has at least one friend or friend of friend, who failed, thus, the expected repayment should be rewritten as

$$\sum_{i=n-5}^{n-1}\sum_{j=\max\{i-2-(n-5),0\}}^{2}\sum_{k=\max\{i-j-(n-5),0\}}^{2}\left[\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}\left(1-\alpha\right)^i\left(1+\frac{j+k\sigma+\left(i-j-k\right)\beta}{n-i}\right)\right].$$

Therefore, the expected repayment should be rewritten as

$$K^r = \sum_{i=0}^{n-1}\sum_{j=\max\{i-2-(n-5),0\}}^{\min\{2,i\}}\sum_{k=\max\{i-j-(n-5),0\}}^{\min\{2,(i-j)\}}\left[\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}\left(1-\alpha\right)^i\left(1+\frac{j+k\sigma+\left(i-j-k\right)\beta}{n-i}\right)\right].$$

Part 3. First we should point out that a line network is similar to a ring network, in which the link between nodes 1 and n is dropped. Thus, for calculating Q^l and K^l, we could use the results of the previous part.

For a line polyad, the chance of game continuation for a line polyad can be written as

$$Q^l = Q^{l_0} + Q^{l_1} + Q^{l_2} + \cdots + Q^{l_{n-1}},$$

in which Q^{l_i} is the probability of a successful group repayment when $i = 1, ..., n-1$ members fail to repay. In such a network, any failing member has at least one and at most two close friends who are willing to pay for him, and at least one and at most two friend of friend who pay for him with probability σ.

When no one fails to repay, the probability of a successful group repayment is $Q^{l_0} = \alpha^n$.

When one member fails, the network will be divided into two sections (similar to the case, in which two members fail in a ring) that consist of h_1 and $h_2 = n - 1 - h_1$ repaying members, respectively. In such a case, each of the two sections repays with probability

$$P(h_i) = \begin{cases} 1 & h_i = 0, 1 \\ \sigma^2 & h_i = 2 \\ \sigma^2 \beta^{2h_i - 4} & h_i \geq 3 \end{cases}.$$

If one of the h_1 or $h_2 = n - 1 - h_1$ is zero, then the failing member must be one of the nodes 1 or n. In such a case, the graph consists of only one repaying section and we simply assume $P(0) = 1$.

If one member defaults, for example in a group of $n = 5$ borrowers, then (h_1, h_2) can become one of the pairs $(0, 4)$, $(1, 3)$, $(2, 2)$, $(3, 1)$, or $(4, 0)$. Therefore, for $n = 5$, when one member defaults, the probability of a successful group repayment will be

$$\alpha^4 (1-\alpha) P(0) P(4) + \alpha^4 (1-\alpha) P(1) P(3) + \alpha^4 (1-\alpha) P(2) P(2) + \alpha^4 (1-\alpha) P(3) P(1) + \alpha^4 (1-\alpha) P(4) P(0).$$

In general, for any $n \geq 5$, if one member fails, then (h_1, h_2) can become $(0, n-1)$, $(1, n-2)$,

$(2, n-3)$, ..., $(n-1, 0)$. In such a case, the probability of a successful group repayment will be

$$Q^{l_1} = \sum_{h=0}^{n-1} \alpha^{n-1} (1-\alpha) P(h_1) P(n-1-h_1).$$

Q^{l_i} can be calculated for any $2 \leq i \leq n-2$ in a similar way.

When $i = n-1$ members fail, the only successful member can be one of the nodes 1 and n or one the nodes 2 and $n-1$ or could be among the other $n-4$ nodes, thus, the probability of a successful group repayment will be

$$\begin{aligned} Q^{l_{n-1}} &= 2\alpha (1-\alpha)^{n-1} \left(1\sigma\beta^{n-3}\right) + 2\alpha (1-\alpha)^{n-1} \left(1^2\sigma\beta^{n-4}\right) \\ &\quad + (n-4)\,\alpha (1-\alpha)^{n-1} \left(1^2\sigma^2\beta^{n-5}\right). \end{aligned}$$

For a line polyad, the expected repayments of nodes 1 and n differ from the ones of the nodes 2 and $n-1$ and differ from the ones of other nodes. Therefore, the average expected repayment of the line network can be written as

$$K^l = \frac{2}{n} K^{l_1} + \frac{2}{n} K^{l_2} + \frac{n-4}{n} K^{l_3},$$

in which K^{l_1}, K^{l_2}, and K^{l_3} represent the expected repayments of nodes 1 and n, of nodes 2 and $n-1$, and of the other nodes, respectively. Below, we calculate K^{l_1}, K^{l_2}, and K^{l_3}.

When i members fail and $2 \leq i \leq n-3$, a successful repaying node 1 or n can have at most one friend and one friend of friend among the failing members, and thus his expected repayment should sum up to

$$\sum_{i=0}^{n-3} \sum_{j=0}^{1} \sum_{k=0}^{1} \binom{1}{j} \binom{1}{k} \binom{n-3}{i-j-k} \alpha^{n-i} (1-\alpha)^i \left(1 + \frac{j + k\sigma + (i-j-k)\beta}{n-i}\right).$$

If $0 \leq i \leq 2$, then a repaying borrower may not have a friend or friend of friend, who failed. Therefore, the expected repayment when $0 \leq i \leq n-3$ should be rewritten as

$$\sum_{i=0}^{n-3} \sum_{j=0}^{\min\{1,i\}} \sum_{k=0}^{\min\{1,(i-j)\}} \binom{1}{j} \binom{1}{k} \binom{n-3}{i-j-k} \alpha^{n-i} (1-\alpha)^i \left(1 + \frac{j + k\sigma + (i-j-k)\beta}{n-i}\right).$$

If $i > n-3$, then a repaying borrower has at least one friend or friend of friend, who failed.

Therefore, if $0 \leq i \leq n-1$ members default, the expected repayment of a repaying node 1 or n should be rewritten as

$$\begin{aligned} K^{l_1} &= \sum_{i=0}^{n-1} \sum_{j=\max\{i-1-(n-3),0\}}^{\min\{1,i\}} \sum_{k=\max\{i-j-(n-3),0\}}^{\min\{1,(i-j)\}} \\ & \left[\binom{1}{j}\binom{1}{k}\binom{n-3}{i-j-k}\alpha^{n-i}(1-\alpha)^i\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right)\right]. \end{aligned}$$

When i members fail, and $3 \leq i \leq n-4$, a successful repaying node 2 or $n-1$ can have at most two friends and one friend of friend among them, and thus his expected repayment should sum up to

$$\sum_{i=0}^{n-4}\sum_{j=0}^{2}\sum_{k=0}^{1}\binom{2}{j}\binom{1}{k}\binom{n-4}{i-j-k}\alpha^{n-i}(1-\alpha)^i\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right).$$

If $0 \leq i \leq 2$, then a repaying borrower 2 or $n-1$ may not have a friend or friend of friend, who fails. Therefore, the expected repayment when $0 \leq i \leq n-4$ should be rewritten as

$$\sum_{i=0}^{n-4}\sum_{j=0}^{\min\{2,i\}}\sum_{k=0}^{\min\{1,(i-j)\}}\binom{2}{j}\binom{1}{k}\binom{n-4}{i-j-k}\alpha^{n-i}(1-\alpha)^i\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right).$$

If $i > n-4$, then a repaying borrower 2 or $n-1$ has at least one friend or friend of friend, who failed.

Thus, if $0 \leq i \leq n-1$ members default, the expected repayment of a repaying borrower 2 or $n-1$ should be rewritten as

$$\begin{aligned} K^{l_2} &= \sum_{i=0}^{n-1} \sum_{j=\max\{i-1-(n-4),0\}}^{\min\{2,i\}} \sum_{k=\max\{i-j-(n-4),0\}}^{\min\{1,(i-j)\}} \\ & \left[\binom{2}{j}\binom{1}{k}\binom{n-4}{i-j-k}\alpha^{n-i}(1-\alpha)^i\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right)\right]. \end{aligned}$$

The expected repayment of any node that is not one of the $1, 2, n-1, n$ can be calculated as follows. When i members fail, and $4 \leq i \leq n-5$, a successful repaying borrower can have at most two friends and two friends of friends among them, and thus his expected repayment should sum up to

$$\sum_{i=0}^{n-5}\sum_{j=0}^{2}\sum_{k=0}^{2}\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}\left(1-\alpha\right)^{i}\left(1+\frac{j+k\sigma+\left(i-j-k\right)\beta}{n-i}\right).$$

If $0\leq i\leq 3$, then a repaying borrower may not have a friend or friend of friend, who fail. Therefore, the expected repayment when $0\leq i\leq n-5$ should be rewritten as

$$\sum_{i=0}^{n-5}\sum_{j=0}^{\min\{2,i\}}\sum_{k=0}^{\min\{2,(i-j)\}}\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}\left(1-\alpha\right)^{i}\left(1+\frac{j+k\sigma+\left(i-j-k\right)\beta}{n-i}\right).$$

If $i>n-5$, then a repaying borrower has at least one friend or friend of friend, who failed. Thus, the expected repayment of a repaying borrower 3, 4, ..., $n-2$ if $0\leq i\leq n-1$ should be rewritten as

$$\begin{aligned}K^{l_3} &= \sum_{i=0}^{n-1}\sum_{j=\max\{i-2-(n-5),0\}}^{\min\{2,i\}}\sum_{k=\max\{i-j-(n-5),0\}}^{\min\{2,(i-j)\}}\\ &\left[\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}\left(1-\alpha\right)^{i}\left(1+\frac{j+k\sigma+\left(i-j-k\right)\beta}{n-i}\right)\right].\ \square\end{aligned}$$

Proof of Lemma 3. 1. 1. For star networks, the chance of game continuation is always decreasing in n, but the expected repayment can be increasing in n if α is not too large.

$$\begin{aligned}Q_{n+1}^{s}\leq Q_{n}^{s}\ \Leftrightarrow\ & \alpha^{n+1}+\sum_{i=1}^{n}\alpha^{n+1-i}\left(1-\alpha\right)^{i}\left[\binom{n}{i}\left(\sigma^{i}\right)^{n-i}+\binom{n}{i-1}\left(\beta^{i-1}\right)^{n+1-i}\right]\\ & \leq\alpha^{n}+\sum_{i=1}^{n-1}\alpha^{n-i}\left(1-\alpha\right)^{i}\left[\binom{n-1}{i}\left(\sigma^{i}\right)^{n-1-i}+\binom{n-1}{i-1}\left(\beta^{i-1}\right)^{n-i}\right],\end{aligned}$$

that can be rewritten as

$$\begin{aligned}Q_{n+1}^{s}\leq Q_{n}^{s}\ \Leftrightarrow\ & \sum_{i=1}^{n}\alpha^{n-i}\left(1-\alpha\right)^{i-1}\left[\binom{n}{i}\left(\sigma^{i}\right)^{n-i}+\binom{n}{i-1}\left(\beta^{i-1}\right)^{n+1-i}\right]\\ & \leq\alpha^{n-1}+\sum_{i=1}^{n-1}\alpha^{n-i-1}\left(1-\alpha\right)^{i-1}\left[\binom{n-1}{i}\left(\sigma^{i}\right)^{n-1-i}+\binom{n-1}{i-1}\left(\beta^{i-1}\right)^{n-i}\right].\end{aligned}$$

If $\alpha \to 1$, for any $i \neq 1$, all the sentences of the left-hand side of the above inequality that include $(1-\alpha)^i$ equal to zero; that is, we will have

$$\begin{aligned} Q^s_{n+1} \leq Q^s_n \;&\Leftrightarrow\; \binom{n}{1}\sigma^{n-1} + \binom{n}{0} \leq 1 + \binom{n-1}{1}\sigma^{n-2} + \binom{n-1}{0} \\ &\Leftrightarrow\; n\sigma^{n-1} - (n-1)\,\sigma^{n-2} \leq 1, \end{aligned}$$

which holds always. If $\alpha \to 0$, for any $i \neq n$, all the sentences of the left-hand side of the above inequality equal to zero, and and for any $i \neq n-1$, all the sentences of the right-hand side of the above inequality equal to zero; that is, we will have

$$\begin{aligned} Q^s_{n+1} \leq Q^s_n \;&\Leftrightarrow\; \binom{n}{n} + \binom{n}{n-1}\left(\beta^{n-1}\right) \leq \binom{n-1}{n-1} + \binom{n-1}{n-2}\beta^{n-2} \\ &\Leftrightarrow\; \beta \leq \frac{n-1}{n}. \end{aligned}$$

which holds almost always (when β is not too large). Moreover,

$$\begin{aligned} K^s_{n+1} \leq K^s_n \;\Leftrightarrow\;& \frac{n}{n+1}\sum_{i=0}^{n-1}\alpha^{n+1-i}(1-\alpha)^i\binom{n-1}{i}\left(1+\frac{i\sigma}{n+1-i}\right) \\ &+\frac{n}{n+1}\sum_{i=1}^{n}\alpha^{n+1-i}(1-\alpha)^i\binom{n-1}{i-1}\left(1+\frac{1+(i-1)\,\beta}{n+1-i}\right) \\ &+\frac{1}{n+1}\sum_{i=0}^{n}\alpha^{n+1-i}(1-\alpha)^i\binom{n}{i}\left(1+\frac{i}{n+1-i}\right) \\ \leq\;& \frac{n-1}{n}\sum_{i=0}^{n-2}\alpha^{n-i}(1-\alpha)^i\binom{n-2}{i}\left(1+\frac{i\sigma}{n-i}\right) \\ &+\frac{n-1}{n}\sum_{i=1}^{n-1}\alpha^{n-i}(1-\alpha)^i\binom{n-2}{i-1}\left(1+\frac{1+(i-1)\,\beta}{n-i}\right) \\ &+\frac{1}{n}\sum_{i=0}^{n-1}\alpha^{n-i}(1-\alpha)^i\binom{n-1}{i}\left(1+\frac{i}{n-i}\right). \end{aligned}$$

If $\alpha \to 1$, all sentences including $(1-\alpha)^i$ equal to zero if $i \neq 0$; that is, we will have

$$\begin{aligned} K^s_{n+1} \leq K^s_n \quad &\Leftrightarrow \quad \frac{n}{n+1}\binom{n-1}{0} + \frac{1}{n+1}\binom{n}{0} \leq \frac{n-1}{n}\binom{n-2}{0} + \frac{1}{n}\binom{n-1}{0} \\ &\Leftrightarrow \quad \frac{n}{n+1} + \frac{1}{n+1} \leq \frac{n-1}{n} + \frac{1}{n}, \end{aligned}$$

which holds always. If $\alpha \to 0$, after canceling out α from both side of the inequality, all sentences of its left-hand side, in which $i \neq n$, equal to zero, and all sentences of its right-hand side, in which $i \neq n-1$, equal to zero; that is, we will have

$$\begin{aligned} K^s_{n+1} \leq K^s_n \quad &\Leftrightarrow \quad \frac{n}{n+1}\binom{n-1}{n-1}\left(1+1+(n-1)\beta\right) + \frac{1}{n+1}\binom{n}{n}(1+n) \\ &\quad \leq \frac{n-1}{n}\binom{n-2}{n-2}\left(1+1+(n-2)\beta\right) + \frac{1}{n}\binom{n-1}{n-1}(1+n-1) \\ &\Leftrightarrow \quad (n+2)(n-1)\beta \leq -2, \end{aligned}$$

which can never hold.

2. For ring networks, the chance of game continuation is always decreasing in n, but the expected repayment can be increasing in n if α is not too large.

$$\begin{aligned} Q^r_{n+1} \leq Q^r_n \quad \Leftrightarrow \quad & \alpha^{n+1} + (n+1)\alpha^n(1-\alpha)\left(\sigma^2\beta^{n-4}\right) \\ & + \sum_{h_1=0}^{n-1}(n+1)\alpha^{n-1}(1-\alpha)^2 P(h_1)P(n-1-h_1) \\ & + \sum_{h_1=0}^{n-2}\sum_{h_2=0}^{n-2-h_1}(n+1)\alpha^{n-2}(1-\alpha)^3 P(h_1)P(h_2)P(n-2-h_1-h_2) \\ & +... + (n+1)\alpha(1-\alpha)^n\left(\sigma^2\beta^{n-4}\right) \\ \leq \quad & \alpha^n + n\alpha^{n-1}(1-\alpha)\left(\sigma^2\beta^{n-5}\right) \\ & + \sum_{h_1=0}^{n-2} n\alpha^{n-2}(1-\alpha)^2 P(h_1)P(n-2-h_1) \\ & + \sum_{h_1=0}^{n-3}\sum_{h_2=0}^{n-3-h_1} n\alpha^{n-3}(1-\alpha)^3 P(h_1)P(h_2)P(n-3-h_1-h_2) \\ & + \cdots + n\alpha(1-\alpha)^{n-1}\left(\sigma^2\beta^{n-5}\right). \end{aligned}$$

If $\alpha \to 1$, the above inequality boils down to

$$Q_{n+1}^r \leq Q_n^r \quad \Leftrightarrow \quad \alpha^{n+1} \leq \alpha^n,$$

which holds always; and if $\alpha \to 0$, the inequality turns to

$$\begin{aligned} Q_{n+1}^r \leq Q_n^r \quad &\Leftrightarrow \quad (n+1)\left(\sigma^2 \beta^{n-4}\right) \leq n\left(\sigma^2 \beta^{n-5}\right) \\ &\Leftrightarrow \quad \beta \leq \frac{n}{n+1}, \end{aligned}$$

which holds if β is not too large.

$$\begin{aligned} K_{n+1}^r \leq K_n^r \quad \Leftrightarrow \quad & \sum_{i=0}^{n} \sum_{j=\max\{i-2-(n-4),0\}}^{\min\{2,i\}} \sum_{k=\max\{i-j-(n-4),0\}}^{\min\{2,(i-j)\}} \\ & \left[\binom{2}{j}\binom{2}{k}\binom{n-4}{i-j-k}\alpha^{n+1-i}(1-\alpha)^i\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n+1-i}\right)\right] \\ & \leq \sum_{i=0}^{n-1} \sum_{j=\max\{i-2-(n-5),0\}}^{\min\{2,i\}} \sum_{k=\max\{i-j-(n-5),0\}}^{\min\{2,(i-j)\}} \\ & \left[\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}(1-\alpha)^i\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right)\right]. \end{aligned}$$

If $\alpha \to 1$, the sentences of the above inequality for any $i \neq 0$ are zero; that is,

$$\begin{aligned} K_{n+1}^r \leq K_n^r \quad \Leftrightarrow \quad & \sum_{j=0}^{0}\sum_{k=0}^{0}\left[\binom{2}{j}\binom{2}{k}\binom{n-4}{-j-k}\alpha^{n+1}\left(1+\frac{j+k\sigma+(-j-k)\beta}{n+1}\right)\right] \\ & \leq \sum_{j=0}^{0}\sum_{k=0}^{0}\left[\binom{2}{j}\binom{2}{k}\binom{n-5}{-j-k}\alpha^{n}\left(1+\frac{j+k\sigma+(-j-k)\beta}{n}\right)\right] \\ \Leftrightarrow \quad & \alpha^{n+1} \leq \alpha^n, \end{aligned}$$

which holds always. If $\alpha \to 0$, all the sentences of the right-hand side of the above inequality for any $i \neq n$ are zero, and all the sentences of the left-hand side of the above inequality for any $i \neq n-1$

are zero (note that one α should be canceled out from both sides); that is,

$$
\begin{aligned}
K_{n+1}^{r} \leq K_{n}^{r} \quad \Leftrightarrow \quad & \sum_{j=2}^{2}\sum_{k=2}^{2}\left[\binom{2}{j}\binom{2}{k}\binom{n-4}{n-j-k}(1-\alpha)^{n}\left(1+j+k\sigma+(n-j-k)\beta\right)\right] \\
& \leq \sum_{j=2}^{2}\sum_{k=2}^{2}\left[\binom{2}{j}\binom{2}{k}\binom{n-5}{n-1-j-k}(1-\alpha)^{n-1}\left(1+j+k\sigma+(n-1-j-k)\beta\right)\right] \\
\Leftrightarrow \quad & \beta \leq 0,
\end{aligned}
$$

which cannot hold except for $\beta = 0$.

3. For a line, the chance of game continuation is always decreasing in n, but the expected repayment can be increasing in n if α is not too large.

$$
\begin{aligned}
Q_{n+1}^{l} \leq Q_{n}^{l} \quad \Leftrightarrow \quad & \alpha^{n+1} + 2\alpha^{n}(1-\alpha)\sigma\beta^{n-2} + 2\alpha^{n}(1-\alpha)\sigma\beta^{n-3} + (n-3)\alpha^{n}(1-\alpha)\sigma^{2}\beta^{n-4} \\
& +\ldots+ \\
& +2\alpha(1-\alpha)^{n}\sigma\beta^{n-2} + 2\alpha(1-\alpha)^{n}\sigma\beta^{n-3} + (n-3)\alpha(1-\alpha)^{n}\sigma^{2}\beta^{n-4} \\
\leq \quad & \alpha^{n} + 2\alpha^{n-1}(1-\alpha)\sigma\beta^{n-3} + 2\alpha^{n-1}(1-\alpha)\sigma\beta^{n-4} + (n-4)\alpha^{n-1}(1-\alpha)\sigma^{2}\beta^{n-5} \\
& +\ldots+ \\
& +2\alpha(1-\alpha)^{n-1}\sigma\beta^{n-3} + 2\alpha(1-\alpha)^{n-1}\sigma\beta^{n-4} + (n-4)\alpha(1-\alpha)^{n-1}\sigma^{2}\beta^{n-5},
\end{aligned}
$$

that can be rewritten as

$$
\begin{aligned}
Q_{n+1}^{l} \leq Q_{n}^{l} \quad \Leftrightarrow \quad & 2\alpha^{n-1}\sigma\beta^{n-2} + 2\alpha^{n-1}\sigma\beta^{n-3} + (n-3)\alpha^{n-1}\sigma^{2}\beta^{n-4} \\
& +\ldots+ \\
& +2(1-\alpha)^{n-1}\sigma\beta^{n-2} + 2(1-\alpha)^{n-1}\sigma\beta^{n-3} + (n-3)(1-\alpha)^{n-1}\sigma^{2}\beta^{n-4} \\
\leq \quad & \alpha^{n-1} + 2\alpha^{n-2}\sigma\beta^{n-3} + 2\alpha^{n-2}\sigma\beta^{n-4} + (n-4)\alpha^{n-2}\sigma^{2}\beta^{n-5} \\
& +\ldots+ \\
& +2(1-\alpha)^{n-2}\sigma\beta^{n-3} + 2(1-\alpha)^{n-2}\sigma\beta^{n-4} + (n-4)(1-\alpha)^{n-2}\sigma^{2}\beta^{n-5}.
\end{aligned}
$$

If $\alpha \to 1$, all the sentences that include $(1-\alpha)$ equal to zero; that is, we will have

$$\begin{aligned} Q_{n+1}^{l} \leq Q_{n}^{l} &\Leftrightarrow 2\sigma\beta^{n-2} + 2\sigma\beta^{n-3} + (n-3)\,\sigma^{2}\beta^{n-4} \\ &\quad \leq 1 + 2\sigma\beta^{n-3} + 2\sigma\beta^{n-4} + (n-4)\,\sigma^{2}\beta^{n-5} \\ &\Leftrightarrow 2\sigma\left(\beta^{n-2} - \beta^{n-4}\right) + \sigma^{2}\left[(n-3)\,\beta^{n-4} - (n-4)\,\beta^{n-5}\right] \leq 1, \end{aligned}$$

which holds always. If $\alpha \to 0$, all the sentences that include α equal to zero; that is, we will have

$$\begin{aligned} Q_{n+1}^{l} \leq Q_{n}^{l} &\Leftrightarrow 2\sigma\beta^{n-2} + 2\sigma\beta^{n-3} + (n-3)\,\sigma^{2}\beta^{n-4} \\ &\quad \leq 2\sigma\beta^{n-3} + 2\sigma\beta^{n-4} + (n-4)\,\sigma^{2}\beta^{n-5} \\ &\Leftrightarrow 2\sigma\left(\beta^{n-2} - \beta^{n-4}\right) + \sigma^{2}\left[(n-3)\,\beta^{n-4} - (n-4)\,\beta^{n-5}\right] \leq 0, \end{aligned}$$

which can hold if β is small enough.

$$\begin{aligned} K_{n+1}^{l} \leq K_{n}^{l} \;\Leftrightarrow\; & \frac{2}{n+1}\sum_{i=0}^{n}\;\sum_{j=\max\{i-1-(n-2),0\}}^{\min\{1,i\}}\;\sum_{k=\max\{i-j-(n-2),0\}}^{\min\{1,(i-j)\}} \\ & \left[\binom{1}{j}\binom{1}{k}\binom{n-2}{i-j-k}\alpha^{n+1-i}\,(1-\alpha)^{i}\left(1+\frac{j+k\sigma+(i-j-k)\,\beta}{n+1-i}\right)\right] \\ & +\frac{2}{n+1}\sum_{i=0}^{n}\;\sum_{j=\max\{i-1-(n-3),0\}}^{\min\{2,i\}}\;\sum_{k=\max\{i-j-(n-3),0\}}^{\min\{1,(i-j)\}} \\ & \left[\binom{2}{j}\binom{1}{k}\binom{n-3}{i-j-k}\alpha^{n+1-i}\,(1-\alpha)^{i}\left(1+\frac{j+k\sigma+(i-j-k)\,\beta}{n+1-i}\right)\right] \\ & +\frac{n-3}{n+1}\sum_{i=0}^{n}\;\sum_{j=\max\{i-2-(n-4),0\}}^{\min\{2,i\}}\;\sum_{k=\max\{i-j-(n-4),0\}}^{\min\{2,(i-j)\}} \\ & \left[\binom{2}{j}\binom{2}{k}\binom{n-4}{i-j-k}\alpha^{n+1-i}\,(1-\alpha)^{i}\left(1+\frac{j+k\sigma+(i-j-k)\,\beta}{n+1-i}\right)\right] \end{aligned}$$

$$
\begin{aligned}
\leq \quad & \frac{2}{n}\sum_{i=0}^{n-1}\ \sum_{j=\max\{i-1-(n-3),0\}}^{\min\{1,i\}}\ \sum_{k=\max\{i-j-(n-3),0\}}^{\min\{1,(i-j)\}} \\
& \left[\binom{1}{j}\binom{1}{k}\binom{n-3}{i-j-k}\alpha^{n-i}(1-\alpha)^{i}\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right)\right] \\
& +\frac{2}{n}\sum_{i=0}^{n-1}\ \sum_{j=\max\{i-1-(n-4),0\}}^{\min\{2,i\}}\ \sum_{k=\max\{i-j-(n-4),0\}}^{\min\{1,(i-j)\}} \\
& \left[\binom{2}{j}\binom{1}{k}\binom{n-4}{i-j-k}\alpha^{n-i}(1-\alpha)^{i}\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right)\right] \\
& +\frac{n-4}{n}\sum_{i=0}^{n-1}\ \sum_{j=\max\{i-2-(n-5),0\}}^{\min\{2,i\}}\ \sum_{k=\max\{i-j-(n-5),0\}}^{\min\{2,(i-j)\}} \\
& \left[\binom{2}{j}\binom{2}{k}\binom{n-5}{i-j-k}\alpha^{n-i}(1-\alpha)^{i}\left(1+\frac{j+k\sigma+(i-j-k)\beta}{n-i}\right)\right].
\end{aligned}
$$

If $\alpha \to 1$, all the sentences for which $i \neq 0$ equal to zero; that is, we will have

$$
K_{n+1}^{l} \leq K_{n}^{l} \quad \Leftrightarrow \quad \frac{2}{n+1}+\frac{2}{n+1}+\frac{n-3}{n+1} \leq \frac{2}{n}+\frac{2}{n}+\frac{n-4}{n},
$$

which holds always. If $\alpha \to 0$, all the sentences of the left-hand side of the above inequality for any $i \neq n$ are zero, and all the sentences of the right-hand side of the above inequality for any $i \neq n-1$ are zero (note that one α should be canceled out from both sides); that is,

$$
\begin{aligned}
K_{n+1}^{l} \leq K_{n}^{l} \quad \Leftrightarrow \quad & \frac{2}{n+1}(5+2\sigma+(2n-5)\beta)+\frac{n-3}{n+1}(3+2\sigma+(n-4)\beta) \\
& \leq \frac{2}{n}(5+2\sigma+(2n-7)\beta)+\frac{n-4}{n}(3+2\sigma+(n-5)\beta) \\
\Leftrightarrow \quad & 4\sigma+\left(n+n^{2}-6\right)\beta \leq -2,
\end{aligned}
$$

which never holds. □

Proof of Lemma 4. We know

$$L \leq \mathscr{L} = \begin{cases} \frac{KY^H}{n(1+\epsilon)} & \text{if} \quad \delta \geq \frac{n}{nQ+n\alpha-k_m} \\ \frac{\alpha\delta KY^H}{(n-n\delta Q+\delta k_m)(1+\epsilon)} & \text{if} \quad \delta \leq \frac{n}{nQ+n\alpha-k_m} \end{cases},$$
$$R = \frac{L(1+\epsilon)}{K},$$
$$V^R = \frac{n\alpha Y^H - nL(1+\epsilon)}{1-\delta Q}.$$

First of all, the borrower welfare V^R is increasing in L because we assumed that a larger loan increases the project outcome more than the repayment; that is, $n\alpha\left(\frac{\partial Y^H}{\partial L}\right) > n(1+\epsilon)$. Moreover, the loan ceiling $\mathscr{L}$ and the borrower welfare V^R are both increasing in Q and K.

Below, we show that the loan ceiling $\mathscr{L}$ and the borrower welfare V^R can be also increasing in n for some δ.

$$\frac{\partial \mathscr{L}}{\partial n} = \begin{cases} \frac{\partial \tilde{L}}{\partial n} = \left(\frac{Y^H}{1+\epsilon}\right) \frac{\frac{\partial K}{\partial n}n - K}{n^2} & \text{if} \quad \delta \geq \frac{n}{nQ+n\alpha-k_m} \\ \frac{\partial \hat{L}}{\partial n} = \left(\frac{\alpha\delta Y^H}{1+\epsilon}\right) \frac{\frac{\partial K}{\partial n}(n-n\delta Q+\delta k_m) - K\left(1-\delta Q - n\delta\frac{\partial Q}{\partial n} + \delta\frac{\partial k_m}{\partial n}\right)}{(n-n\delta Q+\delta k_m)^2} & \text{if} \quad \delta \leq \frac{n}{nQ+n\alpha-k_m} \end{cases},$$
$$\frac{\partial V^R}{\partial n} = \frac{n\left[\alpha\frac{\partial Y^H}{\partial L} - (1+\epsilon)\right]\left(\frac{\partial L}{\partial n}\right)(1-\delta Q) + \left[\alpha Y^H - L(1+\epsilon)\right]\left(1-\delta Q + n\delta\frac{\partial Q}{\partial n}\right)}{(1-\delta Q)^2}.$$

Assuming that $\frac{\partial K}{\partial n} < \frac{K}{n}$, $\tilde{L}$ is always decreasing in n, but $\hat{L}$ can be increasing in n if and only if

$$\frac{\partial K}{\partial n}(n - n\delta Q + \delta k_m) > K\left(1 - \delta Q - n\delta\frac{\partial Q}{\partial n} + \delta\frac{\partial k_m}{\partial n}\right)$$

which can be rewritten as

$$\left[n\left(\frac{K}{n} - \frac{\partial K}{\partial n}\right)Q + \left(n\frac{\partial Q}{\partial n} - \frac{\partial k_m}{\partial n}\right)K + \frac{\partial K}{\partial n}k_m\right]\delta > n\left(\frac{K}{n} - \frac{\partial K}{\partial n}\right).$$

Thus, if $n\left(\frac{K}{n} - \frac{\partial K}{\partial n}\right)Q + \left(n\frac{\partial Q}{\partial n} - \frac{\partial k_m}{\partial n}\right)K + \frac{\partial K}{\partial n}k_m > 0$, then $\hat{L}$ will be increasing in n for any

$$\delta > \frac{n\left(\frac{K}{n} - \frac{\partial K}{\partial n}\right)}{n\left(\frac{K}{n} - \frac{\partial K}{\partial n}\right)Q + \left(n\frac{\partial Q}{\partial n} - \frac{\partial k_m}{\partial n}\right)K + \frac{\partial K}{\partial n}k_m};$$

and if $n\left(\frac{K}{n}-\frac{\partial K}{\partial n}\right)Q+\left(n\frac{\partial Q}{\partial n}-\frac{\partial k_m}{\partial n}\right)K+\frac{\partial K}{\partial n}k_m<0$, $\hat{L}$ will be decreasing in n for any δ.

Therefore, $\mathscr{L}$ can be increasing in n only if $n\left(\frac{K}{n}-\frac{\partial K}{\partial n}\right)Q+\left(n\frac{\partial Q}{\partial n}-\frac{\partial k_m}{\partial n}\right)K+\frac{\partial K}{\partial n}k_m>0$ and

$$\delta\in\left(\frac{n\left(\frac{K}{n}-\frac{\partial K}{\partial n}\right)}{n\left(\frac{K}{n}-\frac{\partial K}{\partial n}\right)Q+\left(n\frac{\partial Q}{\partial n}-\frac{\partial k_m}{\partial n}\right)K+\frac{\partial K}{\partial n}k_m},\frac{n}{nQ+n\alpha-k_m}\right).$$

Moreover, $\frac{\partial V^R}{\partial n}$ can be positive only if

$$n\left[\alpha\frac{\partial Y^H}{\partial L}-(1+\epsilon)\right]\left(\frac{\partial L}{\partial n}\right)(1-\delta Q)+\left(1-\delta Q+n\delta\frac{\partial Q}{\partial n}\right)\left[\alpha Y^H-L(1+\epsilon)\right]>0,$$

which simplifies to

$$\frac{\partial L}{\partial n}>-\left(\frac{1-\delta Q+n\delta\frac{\partial Q}{\partial n}}{n-n\delta Q}\right)\left(\frac{\alpha Y^H-L(1+\epsilon)}{\alpha\frac{\partial Y^H}{\partial L}-(1+\epsilon)}\right).\ \square$$

Proof of Proposition 5. According to Lemma 1, if $\left(K-n\frac{\partial K}{\partial n}\right)Q+\left(n\frac{\partial Q}{\partial n}-\frac{\partial k_m}{\partial n}\right)K+\left(\frac{\partial K}{\partial n}\right)k_m>0$ L can increase in n for given α and δ, there are some n such that

$$\delta\in\left(\frac{K-n\frac{\partial K}{\partial n}}{\left(K-n\frac{\partial K}{\partial n}\right)Q+\left(n\frac{\partial Q}{\partial n}-\frac{\partial k_m}{\partial n}\right)K+\left(\frac{\partial K}{\partial n}\right)k_m},\frac{n}{nQ+n\alpha-k_m}\right).$$

The borrower welfare V^R can increase in n only if L increases in n. Assume

$$\begin{cases}\hat{\delta}(n,\alpha)=\frac{K-n\frac{\partial K}{\partial n}}{\left(K-n\frac{\partial K}{\partial n}\right)Q+\left(n\frac{\partial Q}{\partial n}-\frac{\partial k_m}{\partial n}\right)K+\left(\frac{\partial K}{\partial n}\right)k_m}\\ \tilde{\delta}(n,\alpha)=\frac{n}{nQ+n\alpha-k_m}\end{cases}.$$

For a complete network of borrowers, $Q^c=K^c=k^c_m=[1-(1-\alpha)^n]$ and thus $\frac{\partial Q^c}{\partial n}=\frac{\partial K^c}{\partial n}=\frac{\partial k^c_m}{\partial n}>0$. For such a complete network, we will have

$$\hat{\delta}(n,\alpha)=\frac{[1-(1-\alpha)^n]+(1-\alpha)^n\ln(1-\alpha)^n}{[1-(1-\alpha)^n]^2},$$

$$\tilde{\delta}(n,\alpha)=\frac{1}{\alpha+\frac{n-1}{n}[1-(1-\alpha)^n]}.$$

It is not difficult to verify that $\hat{\delta}$ is strictly increasing in both n and α and $\tilde{\delta}$ is strictly decreasing in both n and α. Moreover, $\lim_{\alpha\to 0}\hat{\delta}\left(n,\alpha\right)=-\infty$ and $\lim_{\alpha\to 0}\tilde{\delta}\left(n,\alpha\right)=\infty$ while $\lim_{\alpha\to 1}\hat{\delta}_{SJL}\left(n,\alpha\right)=1$ and $\lim_{\alpha\to 1}\tilde{\delta}_{SJL}\left(n,\alpha\right)=\frac{n}{2n-1}$. Thus $\hat{\delta}$ and $\tilde{\delta}$ must coincide only once at some critical $\bar{\alpha}\neq 0$. Thus, for any given n, if $\alpha\in(0,\bar{\alpha})$, then $\hat{\delta}<\tilde{\delta}$, and for $\alpha\in(0,\bar{\alpha})$, the loan ceiling will be increasing in n and maximized at

$$N_{\alpha,\delta}=\max\left\{n|\quad \hat{\delta}\left(n,\alpha\right)<\delta<\tilde{\delta}\left(n,\alpha\right)\right\}.$$

Note that n cannot grow too large, as otherwise, $\lim_{n\to\infty}\hat{\delta}=1$ and $\lim_{n\to\infty}\tilde{\delta}=\dfrac{1}{\alpha+1}<1$; that is, for a very large n, the interval $\left(\hat{\delta},\tilde{\delta}\right)$ is empty and the borrower welfare cannot be increased by enlarging the group size. For given α and δ, we can calculate $N_{\alpha,\delta}$ by starting with $n=2$ and increasing n one by one until δ equals one of the $\hat{\delta}$ or $\tilde{\delta}$, and n cannot be increased further. If $\hat{\delta}=\delta$, then $N_{\alpha,\delta}=\left\lfloor\hat{\delta}^{-1}\left(\delta,\alpha\right)\right\rfloor$ and if $\tilde{\delta}=\delta$, then $N_{\alpha,\delta}=\left\lfloor\tilde{\delta}^{-1}\left(\delta,\alpha\right)\right\rfloor$. Thus, $N_{\alpha,\delta}$ can be rewritten as

$$N_{\alpha,\delta}=\min\left\{\left\lfloor\hat{\delta}^{-1}\left(\delta,\alpha\right)\right\rfloor,\left\lfloor\tilde{\delta}^{-1}\left(\delta,\alpha\right)\right\rfloor\right\}.\ \square$$